Questions for PHYS 3220

Chapter 1

Q1.1 What is an antiparticle? Have antiprotons been observed in nature? Have antineutrons? And antihydrogen atoms?

Q1.2 How well has Coulomb's inverse-square law been verified experimentally? Assuming that a deviation form the inverse-square law could be expressed as a difference in the exponent, $2 + \delta$, say, instead of 2, what is the current best upper limit on δ?

Q1.3 There are two kinds of electric charges (positive and negative) and there is only one kind of gravitational "charge" (positive mass). Are there forces in nature that have more than two kinds of "charges"? If so, describe them.

Q1.4 If you use the formula $U = \frac{\epsilon_0}{2} \int E^2 dV$ to compute the energy of a point charge, what value do you obtain? How does the textbook address this result?

Q1.5 How does electronic paper, used in many eBook readers, work?

Chapter 2

Q2.1 Explain carefully and in detail the distinction between (a) the electric potential energy of a system of charges, (b) the electric potential energy of a test charge, and (c) the electric potential due to a system of charges.

Q2.2 The formulas $\phi = \sum q_i/(4\pi\epsilon_0 r_i)$ and $\phi = \int \rho\, dV/(4\pi\epsilon_0 r)$ for computing the electric potential from a charge distribution may not work unless all sources are confined to a region of finite volume. What causes this difficulty? How could one compute the electric potential in such cases?

Q2.3 Figure 2.29(a) is an example of a "concept map," a graphical way of representing relationships between ideas. Research the topic of concept maps, and start drawing your own concept map for the content of this course (you will expand it as the course progresses).

Q2.4 A classic question in introductory electromagnetism classes is the following. You may have seen birds sitting on a high-voltage power line. Why are they not electrocuted? Moreover, would it be safe for you to hold one wire of a high-voltage power line with both hands, letting your body hang down far away from the ground or other objects?

Chapter 3

Q3.1 An important result presented in the textbook is the uniqueness theorem for solutions of Laplace's and Poisson's equations. Succinctly summarize the proof of the uniqueness theorem for Poisson's equation.

Q3.2 The textbook computes the coefficients of capacitance for two large capacitor plates inside a conducting box. After converting the textbook results to the case without the box (i.e., make the box infinitely big), explain how to recover equation (3.15) for the capacitance of a parallel-plate capacitor.

Q3.3 A conducting sphere of 1 cm radius has a capacitance (in vacuum) of approximately 1 pF. What is the exact value of its capacitance in vacuum? Does a parallel-plate capacitor with a plate separation of 1 cm have a capacitance of approximately 1 pF? What about a cylinder of 1 cm radius?

Chapter 4

Q4.1 Purcell & Morin's book is praised for the explanations of electricity and magnetism in matter. In the derivation of Ohm's law for an ionic gas, how does the textbook show that the resistance to the motion due to collisions can be described by a drag force proportional to the average velocity of the ions?

Q4.2 Many introductory books on conduction in metals have a section on the Drude model, and although Purcell & Morin do not mention its name explicitly, they do comment on it. What is the Drude model of conduction in metals? According to the textbook, what is incorrect with this model? What modification did the quantum physicist Sommerfeld introduce?

Q4.3 What is the equivalent resistance of two resistors in parallel? Of two resistors in series? And the equivalent capacitance of two capacitors in parallel? And of two capacitors in series? And the equivalent e.m.f. of two batteries in parallel? And of two batteries in series?

Chapter 5

Q5.1 One of the central concepts in special relativity is the idea of simultaneity. The textbook warns against confusing the "relativity of simultaneity" with the fact that an observer not equally distant from two simultaneous explosions may receive light flashes from them at different times. Elaborate on this difference and give examples of the "relativity of simultaneity."

Q5.2 What is the difference between charge conservation and charge invariance?

Q5.3 The textbook states that 'magnetism is a relativistic aspect of electricity' (where?). Not everybody agrees with this statement. Research the topic and formulate your own opinion. (A possible starting point is the section on this topic in the online Supplements.)

Q5.4 Chapter 5 mentions the following two facts. (1) There are relations between field values in different inertial frames at a specific space-time point. (2) Accelerated charges emit electromagnetic radiation. How does the textbook argue that each these two facts supports the existence of electric and magnetic fields independently of their sources?

Chapter 6

Q6.1 The textbook gives two derivations of $\mu_0 = 1/(\epsilon_0 c^2)$. One takes μ_0 and ϵ_0 as fundamental constants. The other takes ϵ_0 and c as fundamental constants. A third way is to take μ_0 and c as fundamental constants. The latter is the way in the SI system, in which the values of μ_0 and c are not even measured but defined. What is the value of ϵ_0 so defined? And of $k = 1/(4\pi\epsilon_0)$? Give at least 10 significant digits.

Q6.2 A uniformly charged disk rotates about its axis. Since each element of charge is moving, there are circular electric currents that generate a magnetic field. On the other hand, since the surface charge density at a given point is the same at all times, there is no current and thus no magnetic field. Which reasoning is correct?

Q6.3 What is the working principle of the speakers in a sound system?

Q6.4 Which is more real, the magnetic field or the vector potential? Problem 6.18 shows that the magnetic field outside a long solenoid is zero while the vector potential is not zero. Can charged particles that move only outside the solenoid be affected by the magnetic field inside the solenoid? Look up Aharonov-Bohm effect.

Chapter 7

Q7.1 Section 7.5 lists three experiments, named I, II, and III. There is a fourth experiment that can be carried out. The loop on table 2 is undone by pulling the wires in such a way that the rate of decrease of the magnetic flux through the loop is the same as it was in Experiments I, II, and III. Does the galvanometer deflect?

Q7.2 One way Lenz's law is mis-used is by having the induced current create a magnetic field that opposes the inducing magnetic field instead of the change in its flux. Construct and describe an example in which the wrong and the right reasonings give opposite results for the direction of the induced current.

Q7.3 You may have noticed the lines scratched on the road pavement near traffic lights to detect the arrival or presence of a car. If you ride a bicycle, you may have been frustrated when such systems fail to detect you and you wait, it seems, for ever. These systems are called inductive-loop traffic detectors. How do they work? Why do they fail to detect the presence of bicycles, or of light motorbikes?

Q7.4 If you examine the formulas for self-inductance and mutual inductance of specific circuits, you will notice that the inductance depends on geometrical quantities only, like size, shape, number of turns, etc. Can you find a general formula for the inductance that contains geometrical quantities only and thus shows that the above statement is true in general? [Hint: dig into the proof of the reciprocity theorem in Section 7.7.]

Chapter 8

Q8.1 An RLC circuit presents a "resonance," which graphically appears as a peak or bump in the plot of the current vs for example the driving frequency (Fig. 8.11 in the textbook). A resonance is characterized by the position of its peak (in Fig. 8.11, this is the resonant frequency) and the width of the peak, given in section 8.2.6 as the Q factor of the resonance, but very often given as the FWHM=full width at half maximum (how is the latter defined?). Resonances appear in other systems as well, for example mechanical systems and particle physics. Look up examples of resonances. Do graphs of resonances always have the same shape? If not, do they have approximately the same shape around the peak (define 'around')?

Q8.2 The proper or characteristic frequency (or frequencies) of an oscillating system (a harmonic oscillator, a network of harmonic oscillators, an RLC circuit, etc.) is the oscillation frequency of the system when it is isolated. Call it ω_0. Let the system be driven by an external source of frequency ω. In general, is there a resonance (or are there resonances) at specific frequencies? Is there a general relation between the resonance frequencies and the proper frequencies?

Q8.3 Should direct current or alternating current be delivered to homes? Research the "War of Currents' in the 1880s. Would you come to the same choice today? Consider whether direct or alternating currents are better to power LEDs, charge electronic devices and electric cars, and for wireless charging.

Q8.4 In Chapter 8, the textbook introduces the use of complex numbers to represent real physical quantities. Gather your thoughts about it and write down simple and clear prescriptions on how to do it and when it may be useful.

Chapter 9

Q9.1 Faraday discovered that a changing magnetic field generates an electric field. The displacement current term in the Ampere-Maxwell law implies that a changing electric field generates a magnetic field. Why didn't Faraday discover the latter effect? How big is the effect? An order of magnitude estimate is enough.

Q9.2 Purcell and Morin describe how the energy flow in an electric circuit is due to the Poynting vector. How does the electromagnetic energy flow to yield the resistance heating? How does it flow to transport energy from a generator to an appliance? Can the electromagnetic energy flow away from a wire? Can it flow across open space from one part of a circuit to another? Give a few examples and describe the energy flow in them.

Q9.3 Most 3D movies shown in theaters nowadays use a polarized 3D system to create the illusion of three dimensions. How does a polarized 3D system work?

Q9.4 For nonmagnetic optical media, the index of refraction n should be equal to the square root of the relative permittivity κ, i.e., $n = \sqrt{\kappa}$. An examination of Table 1.2 in Fowles shows that this relation seems to fail for some substances, in particular water. Fowles hints at possible reasons. Can you expand on this issue?

Chapter 10

Q10.1 Can you describe (at least qualitatively) the motion of the electric dipoles in Figure 10.9 if released from their positions in the figure?

Q10.2 Electric fields stronger than roughly 10^{11} V/m can tear matter to bits: why? Electric fields stronger than roughly 10^{18} V/m can destroy vacuum (a region where there is no matter): why? (Look up 'Schwinger limit').

Q10.3 Purcell & Morin's textbook is excellent in the description of electricity and magnetism in matter, so it is worth to dig into chapter 10. Ponder about the concepts of microscopic and macroscopic electric fields, draw the lattice of ions in a chunk of salt (Na^+Cl^-), and sketch the field lines of both the microscopic and the macroscopic electric fields. Describe them as if you were explaining them to someone else.

Q10.4 If all the molecules in a water droplet would be polarized and aligned, the electric field strength just outside the drop would exceed 10^{10} V/m, enough to make the air around it glow in a corona discharge. After you check that the previous statement is true, explain why it does not happen.

Chapter 11

Q11.1 For the force $\mathbf{F}$ on a magnetic dipole moment $\mathbf{m}$ in a magnetic field $\mathbf{B}$ the textbook give an example (Problem 11.4) showing that formula (11.22), $\mathbf{F} = (\mathbf{m} \cdot \boldsymbol{\nabla})\mathbf{B}$, is in general incorrect while formula (11.23), $\mathbf{F} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B})$, is in general correct. However, the force $\mathbf{F}_e$ on an electric dipole moment $\mathbf{p}$ in an electric field $\mathbf{E}$ is in general correctly given by $\mathbf{F}_e = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$ and not by $\mathbf{F}_e = \boldsymbol{\nabla}(\mathbf{p} \cdot \mathbf{E})$. Why? [Hint: This is connected to a fundamental difference in the near structure of electric and magnetic dipoles.]

Q11.2 How can one permanently magnetize a piece of iron using an electric current? (Search for 'hysteresis'.) Use the description in the textbook to deepen your understanding of the difference between the magnetic field $\mathbf{B}$ and the magnetic field $\mathbf{H}$.

Geometric optics

Q12.1 Newton deduced the laws of reflection and refraction by treating light as a stream of particles (corpuscles) and assuming that when a corpuscle crosses a boundary it is subject to a force perpendicular to the boundary. In the corpuscular theory of light, light travels faster in water than in air (check it!), opposite to the wave theory of light. Fizeau and Foucault raced to their experiments to find out which is right. How close was the race? Who won? How?

Q12.2 Fowles gives a brief introduction to chromatic aberration and spherical aberration. Do some research and find out if there exist lenses that have no chromatic aberration. What about no spherical aberration? What about neither spherical nor chromatic aberration?

Q12.3 Reflective clothing is often used by joggers, cyclists, workers, etc. to increase their visibility to motorists in the dark. A typical reflective fabric reflects the light from a vehicle's beams back in the same direction it came from, whatever the angle at which the light reaches the fabric. How is this achieved without violating the law of reflection?

Wave optics

Q13.1 Fowles describes the Michelson interferometer and the Twyman-Green interferometer. Other famous interferometers are the Mach-Zehnder interferometer and the Fabry-Perot interferometer. What are they and what are they used for?

Q13.2 In 1818, Fresnel submitted a paper to a competition sponsored by the French Academy where he explained diffraction of light using waves. Poisson, a member of the judging committee who was an ardent critic of the wave theory of light, showed that Fresnel's theory would produce a bright spot at the center of the shadow of a circular opaque obstacle. Absurd, he said. What happened next? Why is this story remembered in the field of optics?

Additional Problems and Exercises for PHYS 3220

Problems

A.1 *General solution of the wave equation* $**$

Show that the function $f(\hat{\mathbf{n}} \cdot \mathbf{r} - ut)$ is a solution of the three-dimensional wave equation $\boldsymbol{\nabla}^2 f = (1/u^2)\, \partial^2 f/\partial t^2$, where $\hat{\mathbf{n}}$ is a unit vector and f is any differentiable function of the single argument $\hat{\mathbf{n}} \cdot \mathbf{r} - ut$). (This is problem 1.3 in Fowler.)

A.2 *Two linear polarizers in series* $**$

Linearly polarized light, traveling horizontally and polarized in the horizontal direction, is sent through a train of two linear polarizers. The first is oriented with its transmission axis at 45 degrees and the second has its transmission axis vertical. Show that the emerging light is linearly polarized in the vertical direction; that is, the plane of polarization has been rotated by 90 degrees. (This is problem 2.12 in Fowler.)

A.3 *Reflectance and transmittance* $**$

The reflectance of a surface is defined as the fraction of the incident light energy that is reflected, and is denoted by R_s and R_p for the s and p polarizations, respectively. Fowles shows that in terms of the coefficients of reflection r_s and r_p,

$$R_s = |r_s|^2, \qquad R_p = |r_p|^2.$$

The transmittance is defined as the fraction of the incident light energy that is transmitted, and is denoted by T_s and T_p for the s and p polarizations, respectively. Show that in terms of the coefficients of transmission t_s and t_p,

$$T_s = \frac{n \cos\phi}{\cos\theta}\, |t_s|^2, \qquad T_p = \frac{n \cos\phi}{\cos\theta}\, |t_p|^2,$$

where (as in Fowles) θ is the angle of incidence, ϕ is the angle of refraction, and n is the relative index of refraction. Then use conservation of energy to prove that in absence of absorption

$$R_s + T_s = 1, \qquad R_t + T_t = 1.$$

A.4 *The paraxial approximation* $**$

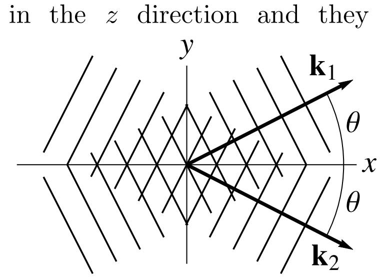
Consider the reflection of a light beam parallel to the axis of a concave spherical mirror of radius R. Let the beam be at a distance d from the axis. The paraxial approximation posits that the reflected beam intersects the axis at the focal point of the mirror. Using the law of reflection without making the paraxial approximation, how far from the focal point does the reflected beam actually intersect the axis? If the paraxial approximation is to be good to 1%, how big can the ratio d/R be?

A.5 *A spatial interference pattern* $**$

Two plane electromagnetic waves of the same angular frequency ω and the same irradiance I_0 move in the xy plane in directions making an angle $+\theta$ and $-\theta$ with the x axis respectively, as shown in the figure. They are linearly polarized in the z direction and they are in phase at the origin. Show that if $\mathbf{k}_1$ and $\mathbf{k}_2$ denote their respective wave vectors, the irradiance at a point $\mathbf{r}$ is given by

$$I(\mathbf{r}) = 4\, I_0\, \cos^2[(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}].$$

From this deduce that the surfaces of constant irradiance are planes parallel to the xz plane separated by a distance $\lambda/(4 \sin\theta)$, where λ is the wave length of each wave.



A.6 *Two long slits* $**$

Two long slits in an opaque screen, 0.10 mm wide with center-to-center separation of 0.21 mm, are illuminated by light from a far away laser with a wavelength of 500 nm. If the plane of observation is 4.0 m away, will the pattern correspond to Fraunhofer or Fresnel diffraction? How many Young's fringes will be seen within the central bright band?

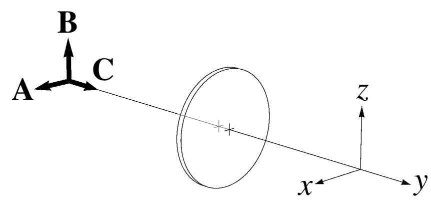
Exercises

A.7 *Brewster angle polarization* $**$

If unpolarized light is incident on a surface at the Brewster angle, the reflected light is linearly polarized and the transmitted light is partially polarized. Show that the degree of polarization of the transmitted light is $P_t = (1 - n^2)^2/(1 + 6n^2 + n^4)$. For glass with $n = 1.5$, this gives $P_t \simeq 8\%$.

A.8 *The image of a vector frame* $**$

Observe the three vectors **A**, **B**, and **C** in the figure with a thin converging lens of focal length f. Each vector has length $0.10f$, and the plane formed by **A** and **B** is at a distance of $1.10f$ from the lens. Describe the image of each vector.



A.9 *Interference from two radio antennas* $**$

Two 1.0-MHz radio antennas emitting in-phase are separated by 600 m along a north-south line. A radio receiver placed 2.0 km east is equidistant from both transmitting antennas and picks up a fairly strong signal. How far north should that receiver be moved if it is again to detect a signal nearly as strong?

A.10 *The smallest spot of light* $**$

No lens can focus light down to a perfect point because there will always be some diffraction. Estimate the size of the minimum spot of light that can be expected at the focus of a lens. Discuss the relationship among the focal length, the lens diameter, and the spot size. Take the f-number of the lens (the ratio of the lens focal length to its diameter) to be roughly 0.8 or 0.9, which is just about what you can expect for a fast lens.

Solutions to Additional Problems for PHYS 3220

A.1 *General solution of the wave equation*

It is easier and clearer to work in Cartesian coordinates. The wave equation reads

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2}. \tag{1}$$

The function we need to verify as solution is

$$f(n_x x + n_y y + n_z z - ut),$$

where n_x, n_y, and n_z are the components of the unit vector $\hat{\mathbf{n}}$, and thus satisfy the relation

$$n_x^2 + n_y^2 + n_z^2 = 1.$$

By the chain rule,

$$\begin{aligned}
\frac{\partial f}{\partial x} &= n_x\, f(n_x x + n_y y + n_z z - ut),\\
\frac{\partial^2 f}{\partial x^2} &= n_x^2\, f(n_x x + n_y y + n_z z - ut).
\end{aligned}$$

Similarly we obtain

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2} &= n_y^2\, f(n_x x + n_y y + n_z z - ut),\\
\frac{\partial^2 f}{\partial z^2} &= n_z^2\, f(n_x x + n_y y + n_z z - ut),\\
\frac{\partial^2 f}{\partial t^2} &= u^2\, f(n_x x + n_y y + n_z z - ut).
\end{aligned}$$

Inserting these expressions into the left and right hand sides of the wave equation (1) gives

$$\text{l.h.s} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (n_x^2 + n_y^2 + n_z^2)\, f = f,$$

and

$$\text{r.h.s} = \frac{1}{u^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{u^2}\, u^2\, f = f.$$

Thus the function $f(\hat{\mathbf{n}}\cdot\mathbf{r} - ut)$ satisfies the wave equation. This is an interesting result because it says that any function that is constant on planes orthogonal to $\hat{\mathbf{n}}$ and moves in the direction of $\hat{\mathbf{n}}$ with speed u is a solution of the wave equation. Thus solutions of the wave equation do not need to be harmonic functions of position.

A.2 *Two linear polarizers in series*

Horizontally polarized light passes first through a polarizer with transmission axis at 45 degrees and then throuhg a polarizer with vertical transmission axis. Let the direction of propagation of the light be $+z$, and let the horizontal and vertical directions be x and y, respectively.

The electric field of the light entering the first polarizer is

$$\mathbf{E}_{\text{in}} = E_0\,\hat{\mathbf{i}}.$$

Only the component of this field along the axis of the first polarizer, which is 45 degrees from the x direction, passes through it. Thus the electric field between the two polarizers is $\mathbf{E}_{\text{middle}} = E_0 \cos(45°)\hat{\mathbf{n}}_{45}$, where $\hat{\mathbf{n}}_{45} = (\hat{\mathbf{i}} + \hat{\mathbf{j}})/\sqrt{2}$ is a unit vector in the direction of the first polarizer axis. So

$$\mathbf{E}_{\text{middle}} = \frac{E_0}{2}\,(\hat{\mathbf{i}} + \hat{\mathbf{j}}).$$

When passing through the second polarizer, only the vertical component of $\mathbf{E}_{\text{middle}}$ is transmitted. So the electric field of the light emerging from the second polarizer is

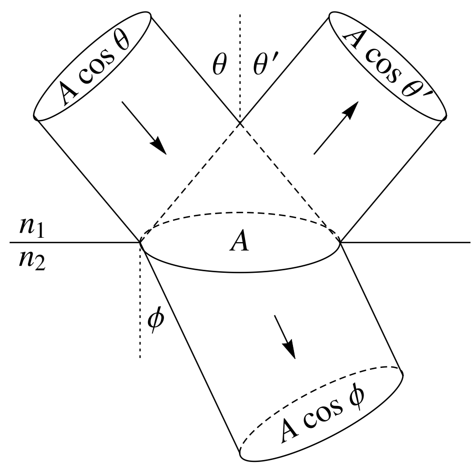
$$\mathbf{E}_{\text{out}} = \frac{E_0}{2}\,\hat{\mathbf{j}}.$$

Thus indeed the polarization of the light has turned by 90 degrees. Notice that its intensity, which is proportional to the square of the magnitude of the electric field, has decreased by a factor of 4 (neglecting the absorption and reflection of light as it passes the polarizers).

If the 45-degree polarizer were missing, the electric field of the light entering the last (and only) polarizer would be $\mathbf{E}_{\text{in}} = E_0\,\hat{\mathbf{i}}$, which has no y component, and no light would emerge from the last polarizer.

A.3 *Reflectance and transmittance*

Consider a circular beam of light incident on a surface area A as in the figure.



The average energy per unit time crossing crossing an area transverse (i.e., *perpendicular*) to the beam is given by the irradiance (W/m^2)

$$I = \frac{n}{Z_0}|E|^2,$$

where $Z_0 = (\mu_0/\epsilon_0)^{1/2}$ is the impedance of the vacuum and E is the amplitude of the electric field. Let I_i, I_r, and I_t be the irradiances of the incident, reflected, and refracted beams. Because of the angle of incidence θ, the transverse area of the incident beam is $A\cos\theta$, so the energy per unit time incident on area A is $P_i = I_i A\cos\theta$. Similarly, the power reflected by area A is $P_r = I_r A\cos\theta'$ and the power refracted is $P_t = I_t A\cos\phi$.

The reflectance is the ratio of the reflected power to the incident power,

$$R = \frac{P_r}{P_i} = \frac{I_r A\cos\theta'}{I_i A\cos\theta} = \frac{I_r}{I_i},$$

since $\theta' = \theta$. Moreover, since the incident and reflected beams are in the same medium, $n_r = n_i = n_1$ and

$$\frac{I_r}{I_i} = \frac{n_r |E_r|^2}{n_i |E_i|^2} = \left| \frac{E_r}{E_i} \right|^2 = |r|^2,$$

where r is r_s for s polarization and r_p for p polarization. Thus we recover the result in Fowles,

$$R = |r|^2.$$

The transmittance is the ratio of the refracted power to the incident power,

$$T = \frac{P_t}{P_i} = \frac{I_t A \cos\phi}{I_i A \cos\theta} = \frac{I_t}{I_i} \frac{\cos\phi}{\cos\theta}.$$

Moreover, the incident and refracted beams are in different media ($n_i = n_1$ and $n_t = n_2$), so

$$\frac{I_t}{I_i} = \frac{n_t |E_t|^2}{n_i |E_i|^2} = n \left| \frac{E_t}{E_i} \right|^2 = n\, |t|^2,$$

where t is t_s for s polarization and t_p for p polarization. Thus we find

$$T = \frac{n \cos\phi}{\cos\theta}\, |t|^2.$$

In the absence of absorption, conservation of energy implies that the amount of energy incident on area A equals the amount of energy reflected by area A plus the amount of energy transmitted through area A, that is

$$P_i = P_r + P_t.$$

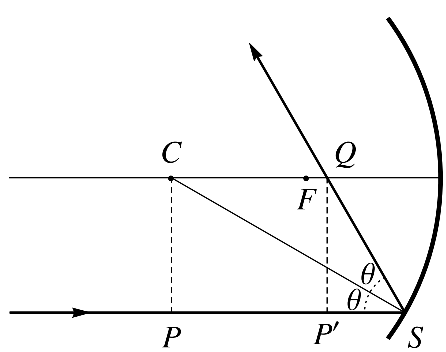
Dividing both sides by P_i and using the definitions of R and T leads immediately to

$$R + T = 1.$$

Since the directions of the s and p polarizations do not change in reflection and refraction, the conservation of energy applies separately to the s and p polarizations.

A.4 *The paraxial approximation*

In the figure below, let C be the center of the mirror, S the point of reflection on the surface of the mirror, and Q the point at which the reflected beam intersects the mirror axis. We want to find the distance FQ between Q and the focal point of the mirror F.



Let P be the point on the incident beam closest to the center of the mirror, and let P' be the point on the incident beam closest to Q. We have

$$CS = R, \quad CP = d, \quad PS = \sqrt{R^2 - d^2}.$$

The angle of incidence is $\theta = \angle PSC$. Thus

$$\tan\theta = \frac{d}{\sqrt{R^2 - d^2}}.$$

By the law of reflection, $\angle PSQ = 2\theta$. Since $P'Q = d$, from the right triangle $P'SQ$ we find

$$P'S = \frac{d}{\tan(2\theta)}.$$

Using $\tan(2\theta) = 2\tan\theta/(1-\tan^2\theta) = 2d\sqrt{R^2-d^2}/(R^2-2d^2)$, we obtain

$$P'S = \frac{R^2 - 2d^2}{2\sqrt{R^2-d^2}}.$$

Finally, the distance QC follows as

$$QC = PS - P'S = \sqrt{R^2-d^2} - \frac{R^2-2d^2}{2\sqrt{R^2-d^2}} = \frac{R^2}{2\sqrt{R^2-d^2}}.$$

We can check that as $d \to 0$ we obtain $QC \to R/2$, which is what we expect from the paraxial approximation. The focal point is on the axis at a distance $R/2$ from both the surface of the mirror and its center. For finite values of d (with $d < R$, otherwise the incident ray does not encounter the mirror), the distance $QC > R/2$, and the reflected ray intersects the axis between the focal point and the mirror. For $d \ll R$, we can expand QC as

$$QC = \frac{R}{2}\frac{1}{\sqrt{1-(d/R)^2}} \simeq \frac{R}{2}\left(1 + \frac{d^2}{2R^2} + \cdots\right).$$

Thus if QC is to be equal to $R/2$ to within 1%, we must have $d^2/2R^2 < 0.01$, that is $d/R < 0.14$. The paraxial approximation is good to a level of 1% for light rays within a distance from the mirror axis of approximately 15% of the mirror radius R.

A.5 *A spatial interference pattern*

It is convenient to write the electric field of each wave in exponential form as

$$\mathbf{E}_1 = \hat{\mathbf{k}}\,E_0\,e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)}, \qquad \mathbf{E}_2 = \hat{\mathbf{k}}\,E_0\,e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)}.$$

The irradiance of each wave is given by $I_0 = E_0^2/Z_0$. The combined wave has electric field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ and the combined irradiance is

$$I(\mathbf{r}) = \frac{1}{Z_0}|\mathbf{E}_1 + \mathbf{E}_2|^2 = I_0\left|e^{i(\mathbf{k}_1\cdot\mathbf{r}-\omega t)} + e^{i(\mathbf{k}_2\cdot\mathbf{r}-\omega t)}\right|^2.$$

Now $|e^{i\alpha} + e^{i\beta}|^2 = |e^{i\alpha}|^2 + |e^{i\beta}|^2 + 2\,\mathrm{Re}(e^{i\alpha}e^{-i\beta}) = 2 + 2\cos(\alpha-\beta) = 4\cos^2(\alpha-\beta)$. So,

$$I(\mathbf{r}) = 4I_0\cos^2[(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r}].$$

The combined irradiance is maximal when the argument of the $\cos^2$ function is equal to an integer multiple of π, i.e.,

$$(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r} = n\pi, \qquad n = 0, \pm1, \pm2, \ldots$$

From the figure, $\mathbf{k}_1 = \hat{\mathbf{i}}\,k\,\cos\theta + \hat{\mathbf{j}}\,k\,\sin\theta$ and $\mathbf{k}_2 = \hat{\mathbf{i}}\,k\,\cos\theta - \hat{\mathbf{j}}\,k\,\sin\theta$, where $k = \omega/c = 2\pi/\lambda$ is the wave number of each plane wave. Thus

$$(\mathbf{k}_1 - \mathbf{k}_2)\cdot\mathbf{r} = \frac{4\pi y\sin\theta}{\lambda}.$$

The condition for maximal combined irradiance therefore becomes

$$y = \frac{n\lambda}{4\sin\theta}, \qquad n = 0, \pm1, \pm2, \ldots$$

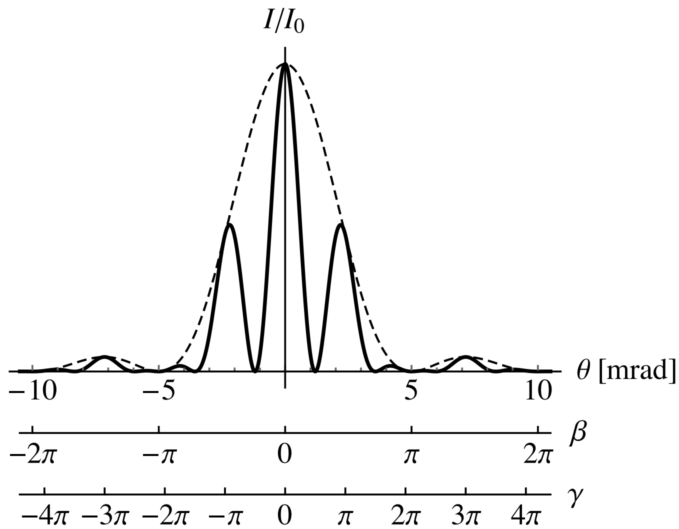
These are the equations of planes parallel to the xz axis separated by a distance $\lambda/(4\sin\theta)$.

A.6 *Two long slits*

Fraunhofer diffraction occurs when the wavefront across the aperture can be considered planar. The criterion for Fraunhofer diffraction in the case of apertures in a flat opaque screen is equation (5.15) in Fowles, namely

$$\frac{1}{2}\left(\frac{1}{d'}+\frac{1}{d}\right)\delta^2 \ll \lambda,$$

where d' and d are the perpendicular distances of the source and observation points from the opaque screen, δ is the width of the aperture, and λ is the wavelength of the light. With the data in the problem ($d' \to \infty$, $d = 4.0$ m, $h = 0.21$ mm, $b = 0.10$ mm, $\delta = h + 2(b/2) = 0.31$ mm), the left hand side is $\delta^2/(2d) = 12.0$ nm, to be compared with $\lambda = 500$ nm. So the pattern corresponds to Fraunhofer diffraction.



Equation (5.27) in Fowles gives the diffraction pattern for a double slit as

$$\frac{I}{I_0} = \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\gamma,$$

where

$$\beta = \frac{1}{2}kb\sin\theta, \qquad \gamma = \frac{1}{2}kh\sin\theta,$$

with $k = 2\pi/\lambda$. The figure shows a plot of this diffraction pattern with the data in the problem. The factor $[\sin(\beta)/\beta]^2$, which depends on the slit width b, is the diffraction pattern of one slit (dashed line in the figure). The factor $\cos^2\gamma$, which depends on the slit separation h, is the interference factor (not shown in the figure). We count 5 Young's interference maxima within the central bright region $-\pi < \beta < \pi$. Notice that the interference fringes near $\beta = \pm\pi$ are suppressed by the minimum of the diffraction envelope (dashed line). Actually each fringe at $\beta = \pm\pi$ splits into two "half-fringes."

If we want a formula for the number of Young's fringes in the central bright region, we could count the maxima of the diffraction pattern $I(\theta)$ that fall within $-\pi < \beta < \pi$. The maxima of $I(\theta)$, however, are not regularly spaced, and it is more convenient to count the zeros of $I(\theta)$ and then deduce the number of fringes. The positive zeros of the interference factor $\cos^2\gamma$ are at $\gamma = \frac{1}{2}\pi, \frac{3}{2}\pi, \frac{5}{2}\pi, \ldots$, while the first positive zero of the diffraction pattern $[\sin(\beta)/\beta]^2$ is at $\beta = \pi$. The latter corresponds to $\gamma = (h/b)\pi$, since $\gamma/\beta = h/b$. We want to count how many half-integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ lie between 0 and h/b excluded. Let this number be n_0. Then the number of Young's fringes follows as $1 + 2n_0$, since there are n_0 fringes on each side of the central fringe. To find n_0, let $\nu(x)$ be the number of odd integers in the open interval $(0, x)$. Then $n_0 = \nu(2h/b)$. A moment of thought shows that $\nu(x) = \lceil (x+1)/2 \rceil - 1$, where $\lceil y \rceil$ is the largest integer greater or equal to y. So the number of Young's fringes within the central bright region is $2\lceil (h/b) + (1/2) \rceil - 1$.