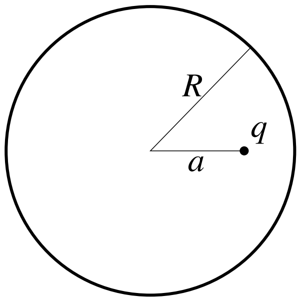
A charge inside a shell

Solve the electrostatic problem of a point charge q inside an uncharged conducting spherical shell of radius R at a distance a from the center of the shell (Figure 1). In particular, find the electric force on the charge q.

Figure 1.



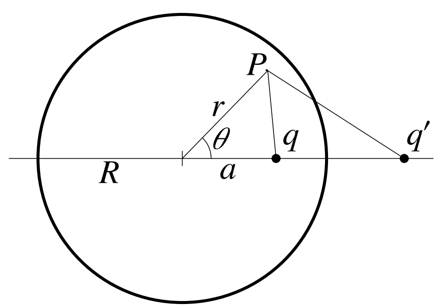
Solution.

The charge q induces a charge distribution on the inner and outer surfaces of the conducting shell. The electric field due to the induced charges exerts a force on the charge q. Thus we proceed by first finding the induced charges and then determining the electric field they generate.

To find the induced charge densities σ_{in} and σ_{out}, and the net charge density $\sigma = \sigma_{\text{in}} + \sigma_{\text{out}}$, we compute the total electric field (charge q plus induced charges) just inside and just outside the shell, which we expect to be perpendicular to the shell, i.e., radial. We do it by computing the total potential and taking its gradient. Then we use $\sigma_{\text{in}} = -\epsilon_0 E_{r,\text{in}}$ and $\sigma_{\text{out}} = \epsilon_0 E_{r,\text{out}}$, where $E_{r,\text{in}}$ and $E_{r,\text{out}}$ are the radial components of the electric field and the signs follow from using small Gaussian surfaces across the inner and outer surfaces of the shell.

To find the electric field due to the induced charges, we subtract the potential due to the charge q from the total potential found earlier and then take the gradient. Multiplying the induced electric field by q gives us the force on the charge q.

Figure 2. Image charge.



Total potential ϕ. We introduce spherical coordinates (r, θ, φ) with origin at the center of the shell and z axis in the direction of charge q (Figure 2). We will find the electric potential inside and outside the spherical shell, setting the potential to zero at infinity.

Outside the shell, we seek a potential function that is constant on the shell. One such potential function is

$$\phi_{\text{out}}(r,\theta,\varphi) = \frac{q}{4\pi\epsilon_0 r}. \tag{1}$$

it is correctly normalized as we can verify by taking a spherical Gaussian surface of radius larger than R and recognizing that the net charge inside of it is q. By the uniqueness theorem, equation (1) is the solution outside the shell.

To find the electric potential at the point $P(r,\theta,\varphi)$ inside the shell ($r \le R$) we use the method of images. We introduce a fictitious charge q' at a distance a' from the center of the shell as in Figure 2 ($a' \ge R$). Then the potential at P is

$$\phi_{\text{in}}(r,\theta,\varphi) = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} + \frac{q'}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a'^2-2a'r\cos\theta}}. \tag{2}$$

We choose q' and a' so that the potential on the shell is constant, in particular equal to zero just for the sake of finding q' and a' (we add the $q/(4\pi\epsilon_0 R)$ potential of the shell later). On the intersections of the shell with the z axis, this gives

$$\begin{cases} \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{R-a} + \dfrac{q'}{4\pi\epsilon_0}\dfrac{1}{a'-R} = 0, \\ \dfrac{q}{4\pi\epsilon_0}\dfrac{1}{R+a} + \dfrac{q'}{4\pi\epsilon_0}\dfrac{1}{R+a'} = 0. \end{cases} \tag{3}$$

Solving the first equation for $q' = q(R-a')/(R-a)$, substituting into the second equation, and bringing the expression to common denominator leads to

$$\begin{aligned} &(R-a)(R+a') + (R-a')(R+a) = 0, \\ \Longrightarrow\ &R(R-a+R+a) + (R-a-R-a)a' = 0, \\ \Longrightarrow\ &2R^2 - 2aa' = 0, \\ \Longrightarrow\ &aa' = R^2, \\ \Longrightarrow\ &a' = \frac{R^2}{a}. \end{aligned} \tag{4}$$

And then

$$q' = q\frac{R-a'}{R-a} = q\frac{R-(R^2/a)}{R-a} = q\frac{aR-R^2}{a(R-a)} = q\frac{R(a-R)}{a(R-a)} = -q\frac{R}{a}. \tag{5}$$

Thus the potential inside the shell is (after adding back the $q/(4\pi\epsilon_0 R)$ potential of the shell)

$$\begin{aligned} \phi_{\text{in}}(r,\theta,\varphi) &= \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{q}{4\pi\epsilon_0}\frac{R}{a}\frac{1}{\sqrt{r^2+\frac{R^4}{a^2}-2\frac{R^2 r}{a}\cos\theta}} + \frac{q}{4\pi\epsilon_0 R} \\ \Longrightarrow \phi_{\text{in}}(r,\theta,\varphi) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2ar\cos\theta}} - \frac{R}{\sqrt{a^2r^2+R^4-2R^2ar\cos\theta}} + \frac{1}{R}\right]. \end{aligned} \tag{6}$$

Figure 3 shows a sketch of the equipotential lines for the total electric potential

$$\phi(r,\theta,\varphi) = \begin{cases} \phi_{\rm in}(r,\theta,\varphi), & r \le R, \\ \phi_{\rm out}(r,\theta,\varphi), & r \ge R. \end{cases} \tag{7}$$

Figure 3 has been produced using the `ContourPlot` function in Mathematica on the function $\phi(r,\theta,0)/\phi_R$ with $\phi_R = q/(4\pi\epsilon_0 R)$ and $a = 0.65R$.

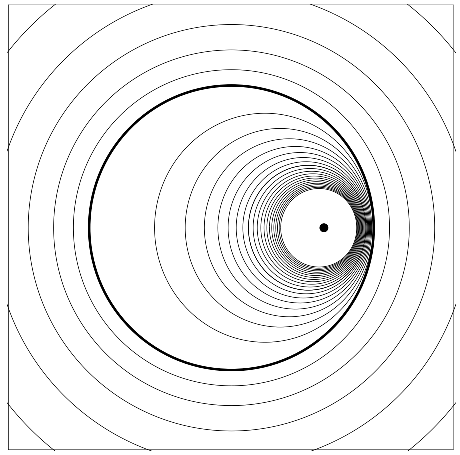


Figure 3. Total electric potential.

Electric field. The spherical components of the electric field outside the shell are the same as those of a Coulomb field,

$$E_{r,\rm out}(r,\theta,\varphi) = \frac{q}{4\pi\epsilon_0 r^2}, \tag{8}$$

$$E_{\theta,\rm out}(r,\theta,\varphi) = 0, \tag{9}$$

$$E_{\varphi,\rm out}(r,\theta,\varphi) = 0. \tag{10}$$

The spherical components of the electric field inside the shell can be obtained from the formula $\mathbf{E} = -\boldsymbol{\nabla}\phi$,

$$E_{r,\rm in}(r,\theta,\varphi) = -\frac{\partial \phi_{\rm in}}{\partial r} = \frac{q}{4\pi\epsilon_0}\left[\frac{r - a\cos\theta}{(r^2+a^2-2ar\cos\theta)^{3/2}} - \frac{aR(ar - R^2\cos\theta)}{(a^2r^2+R^4-2R^2ar\cos\theta)^{3/2}}\right], \tag{11}$$

$$E_{\theta,\rm in}(r,\theta,\varphi) = -\frac{1}{r}\frac{\partial \phi_{\rm in}}{\partial \theta} = \frac{q}{4\pi\epsilon_0}\left[\frac{a\sin\theta}{(r^2+a^2-2ar\cos\theta)^{3/2}} - \frac{aR^3\sin\theta}{(a^2r^2+R^4-2R^2ar\cos\theta)^{3/2}}\right] \tag{12}$$

$$E_{\varphi,\rm in}(r,\theta,\varphi) = -\frac{1}{r\sin\theta}\frac{\partial \phi_{\rm in}}{\partial \varphi} = 0. \tag{13}$$

Figure 4 shows a sketch of the field lines for the total electric field. The field lines outside the shell are radial as for a Coulomb field. The field lines inside the shell are obtained parametrically by numerically solving the equations $dr'/d\lambda = r'^2 E_{r'}, d\theta'/d\lambda = r'^2 E_{\theta'}$ in Mathematica, where (r', θ', φ') are spherical coordinates with the origin on the charge q..

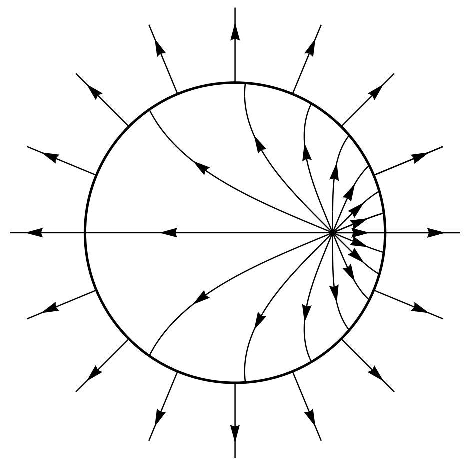


Figure 4. Electric field lines (assuming a positive charge q; reverse the direction of the arrows for a negative charge q).

Charge density σ. The spherical components of the electric field on the outer surface of the shell are obtained setting $r = R$ in Equations (8–10),

$$E_{r,\text{out}}\big|_{r=R} = \frac{q}{4\pi\epsilon_0 R^2}, \tag{14}$$

$$E_{\theta,\text{out}}\big|_{r=R} = 0, \tag{15}$$

$$E_{\varphi,\text{out}}\big|_{r=R} = 0. \tag{16}$$

Thus the surface charge density on the outer surface of the shell is

$$\sigma_{\text{out}} = \epsilon_0 E_{r,\text{out}} = \frac{q}{4\pi R^2}. \tag{17}$$

It is a uniform charge distribution of the same sign as q. Its integral over the surface of the shell is equal to q.

The spherical components of the electric field on the inner surface of the shell are obtained setting $r = R$ in Equations (11–13),

$$\begin{aligned} E_{r,\text{in}}\big|_{r=R} &= \frac{q}{4\pi\epsilon_0}\left[\frac{R - a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} - \frac{a(a - R\cos\theta)}{R(R^2 + a^2 - 2aR\cos\theta)^{3/2}}\right] \\ &= \frac{q}{4\pi\epsilon_0 R}\frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}}, \end{aligned} \tag{18}$$

$$E_{\theta,\text{in}}\big|_{r=R} = \frac{q}{4\pi\epsilon_0 R}\left[\frac{aR\sin\theta}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} - \frac{aR\sin\theta}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}}\right] = 0, \tag{19}$$

$$E_{\varphi,\text{in}}\big|_{r=R} = 0. \tag{20}$$

The electric field on the inner surface is radial as expected. The surface charge density on the inner surface of the shell is

$$\sigma_{\text{in}} = -\epsilon_0 E_{r,\text{in}} = -\frac{q}{4\pi R}\frac{R^2 - a^2}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}}. \tag{21}$$

It is a non-uniform charge distribution of sign opposite to that of q. The inner charge density is stronger on the side of the shell nearest to the charge q. One can check that the integral of σ_{in} over the surface of the shell is equal to $-q$,

$$\int \sigma_{\text{in}} \sin\theta \, d\theta \, d\varphi = -q.$$

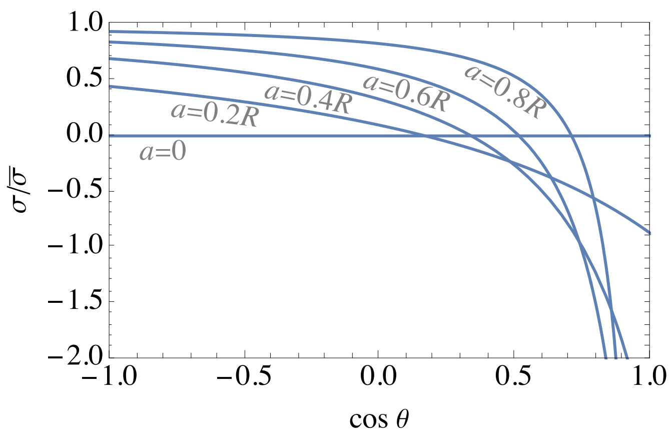
The net surface charge density on the shell finally follows as

$$\sigma = \sigma_{\text{out}} + \sigma_{\text{in}}, \tag{22}$$

$$\Longrightarrow \sigma = \frac{q}{4\pi R^2}\left[1 - \frac{R(R^2-a^2)}{(R^2+a^2-2aR\cos\theta)^{3/2}}\right]. \tag{23}$$

The charge density σ, divided by the average charge density $\overline{\sigma} = q/(4\pi R^2)$, is plotted as a function of $\cos\theta$ in Figure 5 for various values of a. The charge on the shell is of opposite sign to q on the side of the shell closest to q and of the same sign as q on the side of the shell farthest from q.

Figure 5. Induced charge density.



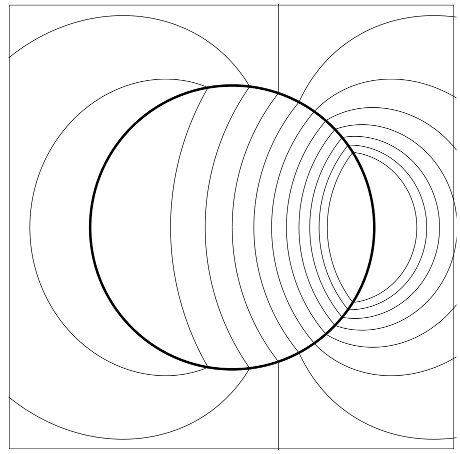
Induced electric potential. The potential due to the induced charges on the shell is obtained from $\phi(r,\theta,\varphi)$ by subtracting the potential due to the point charge at q. Thus

$$\phi_{\text{induced}}(r,\theta,\varphi) = \begin{cases} \dfrac{q}{4\pi\epsilon_0}\left[-\dfrac{R}{\sqrt{a^2r^2+R^4-2R^2ar\cos\theta}} + \dfrac{1}{R}\right], & r \le R, \\ \dfrac{q}{4\pi\epsilon_0}\left[\dfrac{1}{r} - \dfrac{1}{\sqrt{r^2+a^2-2ar\cos\theta}}\right], & r \ge R. \end{cases} \tag{24}$$

The induced potential inside is the same as the potential due to the image charge $q' = -qR/a$ in Figure 2 (apart from the constant $q/(4\pi\epsilon_0 R)$). The induced potential outside the shell is the potential of two equal and opposite charges, a charge $+q$ at the origin and a charge $-q$ at the location of the charge q.

Figure 6 shows equipotential lines for the induced potential, produced with `ContourPlot` function in Mathematica on the function $\phi_{\text{induced}}(r,\theta,0)/\phi_R$ with $\phi_R = q/(4\pi\epsilon_0 R)$ and $a = 0.65R$. .

Figure 6. Induced electric potential.



Induced electric field. The spherical components of the induced electric field are

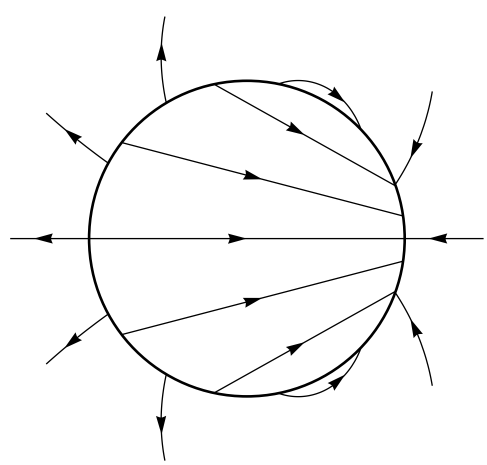
$$E_{r,\text{induced}} = -\frac{\partial \phi_{\text{induced}}}{\partial r} = \begin{cases} -\dfrac{q}{4\pi\epsilon_0} \dfrac{a^2 R r - a R^2 \cos\theta)}{(a^2 r^2 + R^4 - 2R^2 a r \cos\theta)^{3/2}}, & r \leq R, \\ \dfrac{q}{4\pi\epsilon_0 r^2} - \dfrac{q}{4\pi\epsilon_0} \dfrac{r - a\cos\theta}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}}, & r \geq R, \end{cases} \tag{25}$$

$$E_{\theta,\text{induced}} = -\frac{1}{r}\frac{\partial \phi_{\text{induced}}}{\partial \theta} = \begin{cases} -\dfrac{q}{4\pi\epsilon_0} \dfrac{a R^3 \sin\theta}{(a^2 r^2 + R^4 - 2R^2 a r \cos\theta)^{3/2}}, & r \leq R, \\ -\dfrac{q}{4\pi\epsilon_0} \dfrac{a \sin\theta}{(r^2 + a^2 - 2ar\cos\theta)^{3/2}}, & r \geq R, \end{cases} \tag{26}$$

$$E_{\varphi,\text{induced}} = -\frac{1}{r\sin\theta}\frac{\partial \phi_{\text{induced}}}{\partial \varphi} = 0. \tag{27}$$

Figure 7 shows a sketch of the field lines for the induced electric field. The field lines inside the shell are radial as for a Coulomb field centered at position of the image charge. The field lines inside the shell are obtained parametrically by numerically solving the equations $dz/d\lambda = r^3 E_z, dy/d\lambda = r^3 E_y$ in Mathematica, where (x, y, z) are Cartesian coordinates with the origin at the center of the shell.

Figure 7. Induced electric field lines (assuming a positive charge q; reverse the direction of the arrows for a negative charge q).



Electric force on charge q. The force on charge q can be obtained from $\mathbf{F} = q\mathbf{E}_{\text{induced}}$ with $\mathbf{E}_{\text{induced}}$ evaluated at the location of charge q. We compute

$$\begin{aligned} E_{r,\text{induced}}\big|_{r=a,\theta=0} &= \frac{q}{4\pi\epsilon_0}\left[-\frac{aR(a^2-R^2)}{(a^4+R^4-2R^2a^2)^{3/2}}\right] \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{aR(R^2-a^2)}{(R^2-a^2)^3}\right] \\ &= \frac{q}{4\pi\epsilon_0}\frac{aR}{(R^2-a^2)^2}, \end{aligned} \tag{28}$$

$$E_{\theta,\text{induced}}\big|_{r=a,\theta=0} = 0, \tag{29}$$

$$E_{\varphi,\text{induced}}\big|_{r=R} = 0. \tag{30}$$

Thus

$$F_r = \frac{q^2}{4\pi\epsilon_0}\frac{aR}{(R^2-a^2)^2}, \qquad F_\theta = F_\varphi = 0. \tag{31}$$

Figure 8 shows the electric force F_r as a function of the off-center distance a. In the figure, $F_0 = q^2/(4\pi\epsilon_0 R^2)$.

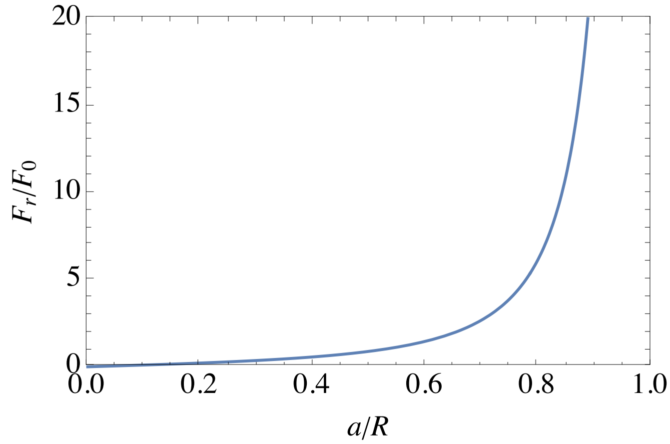


Figure 8. Radial component of the electric force on charge q.

The force is radial and directed outward. Since the force depends on q^2, the force is independent of the sign of the charge q. The charge q is attracted toward the point on the shell closest to q.