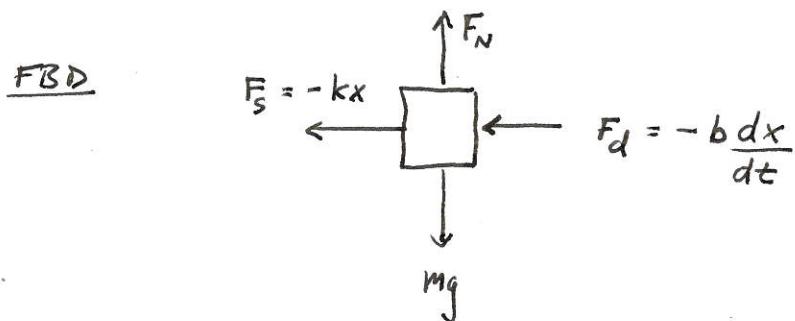
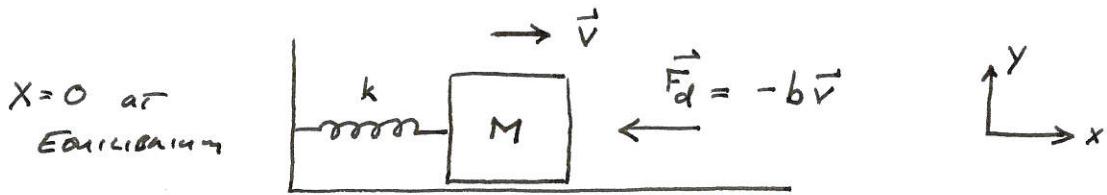


## CH 11: The Damped Harmonic Oscillator

⇒ To good ol' mass & spring, ADD A Velocity Dependent Damping Force



$$\sum F_x = -b \frac{dx}{dt} - kx = m \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0$$

let  $\frac{k}{m} = \omega^2$  as with S.H.O.

$$\frac{b}{m} = 28$$

$$\frac{d^2x}{dt^2} + 28 \frac{dx}{dt} + \omega^2 x = 0$$

"Damped Harmonic  
Oscillator Equation"

$\Rightarrow$  DHO Equation : Guess solution of the form

$$x(t) = A e^{\alpha t} \quad \text{not plug in}$$

$$\frac{dx}{dt} = \alpha A e^{\alpha t} = \alpha x$$

$$\frac{d^2x}{dt^2} = \alpha^2 A e^{\alpha t} = \alpha^2 x$$

Then

$$\alpha^2 x + 2\gamma\alpha x + \omega^2 x = 0$$

$$\alpha^2 + 2\gamma\alpha + \omega^2 = 0$$

Find  $\alpha$  using Quadratic Equation

$$\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2}$$

$$\alpha_1 = -\gamma + \sqrt{\gamma^2 - \omega^2}$$

$$\alpha_2 = -\gamma - \sqrt{\gamma^2 - \omega^2}$$

And

$$x(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t} \quad \text{Two constants of Integration}$$
$$= e^{-\gamma t} \left[ A e^{(\gamma^2 - \omega^2)^{1/2} t} + B e^{-(\gamma^2 - \omega^2)^{1/2} t} \right]$$

Three Possibilities ①  $\gamma^2 > \omega^2$  "Over Damped"

②  $\gamma^2 = \omega^2$  "Critically Damped"

③  $\gamma^2 < \omega^2$  "Under Damped"

i - Overdamped  $\gamma^2 > \omega^2$

recall  $\gamma = \frac{1}{2} \frac{b}{m}$

Define  $\Omega^2 = \gamma^2 - \omega^2$

If  $\Omega^2$  positive,  $\Omega$  is real and

$$x(t) = A e^{-(\gamma-\Omega)t} + B e^{-(\gamma+\Omega)t}$$

Solve A, B from initial conditions ...

Ex Overdamped oscillation released from rest @  $x_0$ , @  $t=0$

$$x(0) = A + B = x_0$$

$$v(0) = -(\gamma - \Omega)A - (\gamma + \Omega)B = 0$$

$$A = -\frac{(\gamma + \Omega)}{(\gamma - \Omega)} B$$

so  $\left(1 - \frac{\gamma + \Omega}{\gamma - \Omega}\right)B = -\frac{2\Omega}{\gamma - \Omega} B = x_0$

$$B = x_0 \left( \frac{\gamma - \Omega}{2\Omega} \right)$$

$$A = x_0 \left( \frac{\Omega + \gamma}{2\Omega} \right)$$

# Tracey

$$x(t) = x_0 \left[ \frac{\Omega + \gamma}{2\Omega} e^{-(\gamma - \Omega)t} + \frac{\Omega - \gamma}{2\Omega} e^{-(\gamma + \Omega)t} \right]$$

$\Omega = \sqrt{\gamma^2 - \omega^2} < \gamma$ , so always exponential decay

(ii) Circular Damped  $\delta^2 = \omega^2$

Only one solution  $X(t) = Ae^{-\delta t}$

But this is incomplete! 2<sup>nd</sup> order DEQ  $\Rightarrow$  2 constants of integration.

Try  $x = u(t) e^{-\delta t}$

$$\dot{x} = \dot{u}e^{-\delta t} - u\delta e^{-\delta t}$$

$$\ddot{x} = \ddot{u}e^{-\delta t} - 2\dot{u}\delta e^{-\delta t} + u\delta^2 e^{-\delta t}$$

Plug back into EoN

$$\cancel{\ddot{u}e^{-\delta t}} - 2\dot{u}\delta e^{-\delta t} + \cancel{u\delta^2 e^{-\delta t}} + \cancel{2\dot{u}\delta e^{-\delta t}} - \cancel{2u\delta^2 e^{-\delta t}} + \cancel{\omega^2 u e^{-\delta t}} = 0$$

$$\ddot{u} - 2\dot{u}\delta + \cancel{u\delta^2} + \cancel{2\dot{u}\delta} - \cancel{2u\delta^2} + \cancel{\omega^2 u} = 0$$

$$\delta^2 = \omega^2$$

$$\ddot{u} = 0, \quad u = Ax + B$$

$$\therefore \boxed{X(t) = (Ax + B)e^{-\delta t}}$$

Find A & B from  
initial conditions,  
is usage.

(iii) Underdamped  $\delta^2 < \omega^2$

Process: We get negative # under square root!

Some math: Taylor Series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Introduce imaginary number  $i = \sqrt{-1}$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i, \text{ etcetera}$$

Then

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \cos \theta + i \sin \theta$$

$$\text{Also } \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$e^{i\theta} + 1 = 0, \text{ etc}$$

Back to underdamped... if  $\gamma^2 < \omega^2 \rightarrow \sqrt{\gamma^2 - \omega^2}$

$$\begin{aligned}
 &= \sqrt{(-1)(\omega^2 - \gamma^2)} \\
 &= i\sqrt{\omega^2 - \gamma^2} \\
 &= i\tilde{\omega} \\
 &\quad \uparrow \text{real number}
 \end{aligned}$$

And

$$x(t) = e^{-\gamma t} (A e^{i\tilde{\omega}t} + B e^{-i\tilde{\omega}t})$$

$$= e^{-\gamma t} [A \cos \tilde{\omega}t + i A \sin \tilde{\omega}t + B \cos \tilde{\omega}t - i B \sin \tilde{\omega}t]$$

$$= e^{-\gamma t} [(A+B) \cos \tilde{\omega}t + i(A-B) \sin \tilde{\omega}t]$$

$$\left. \begin{array}{l} \text{Trig Identity} \\ \cos \alpha \cos \beta - \sin \alpha \sin \beta = \cos(\alpha + \beta) \end{array} \right\}$$

$$x(t) = C e^{-\gamma t} \cos(\tilde{\omega}t + \phi)$$

Graph

