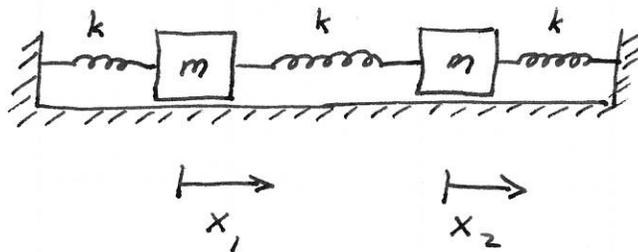


CH6: Coupled Oscillations, Normal Modes

< Demos: Double Pendulum, Coupled Pendula >

- In general, might expect motion to be complex / chaotic
- What can we do?

Ex



$$K = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$U = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k x_2^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k (x_2 - x_1)^2 - \frac{1}{2} k x_2^2$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \right) - \frac{\partial \mathcal{L}}{\partial x_1} = m \ddot{x}_1 + k x_1 - k (x_2 - x_1) = 0$$

$$m \ddot{x}_1 + 2k x_1 - k x_2 = 0$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2} \right) - \frac{\partial \mathcal{L}}{\partial x_2} = m \ddot{x}_2 + 2k x_2 - k x_1 = 0$$

⇒ Neglecting "crossterm" we have 2 independent oscillators

$$x_i = A_i \cos(\omega t + \phi_i) \quad \omega = \sqrt{\frac{k}{m}}$$

⇒ To include cross term, try assuming we still have harmonic solutions ...

$$\ddot{X}_1 + 2\omega^2 X_1 - \omega^2 X_2 = 0$$

$$\ddot{X}_2 + 2\omega^2 X_2 - \omega^2 X_1 = 0$$

$$\text{Sum: } (\ddot{X}_1 + \ddot{X}_2) + \omega^2 (X_1 + X_2) = 0$$

$$\text{Diff: } (\ddot{X}_1 - \ddot{X}_2) + 3\omega^2 (X_1 - X_2) = 0$$

$$\text{So... } X_+ = X_1 + X_2 = A_+ \cos(\omega t + \phi_+)$$

$$X_- = X_1 - X_2 = A_- \cos(\sqrt{3}\omega t + \phi_-)$$

And

$$X_1(t) = \frac{A_+}{2} \cos(\omega t + \phi_+) + \frac{A_-}{2} \cos(\sqrt{3}\omega t + \phi_-)$$

$$X_2(t) = \frac{A_+}{2} \cos(\omega t + \phi_+) - \frac{A_-}{2} \cos(\sqrt{3}\omega t + \phi_-)$$

"Normal Modes" - Set constants of integration A_+ , A_-
Equal to zero, one at a time.

< Overhead: Normal Modes for this system >

⇒ ARBITRARY STATE OF MOTION OF THE SYSTEM CAN BE UNDERSTOOD AS A SUM OF NORMAL MODES. EVALUATE CONSTANTS BY APPLYING INITIAL CONDITIONS.

e.g.

$$x_1(t=0) = 0.3 \text{ m}$$

$$\dot{x}_1(t=0) = 0.0 \text{ m/s}$$

$$x_2(t=0) = 0$$

$$\dot{x}_2(t=0) = 0$$

$$\textcircled{i} \quad x_1(0) = \frac{A_+}{2} \cos \phi_+ + \frac{A_-}{2} \cos \phi_- = 0.3 \text{ m}$$

$$\textcircled{ii} \quad \dot{x}_1(0) = -\frac{A_+}{2} \omega \sin \phi_+ - \frac{A_-}{2} \sqrt{3} \omega \sin \phi_- = 0$$

$$\textcircled{iii} \quad x_2(0) = \frac{A_+}{2} \cos \phi_+ - \frac{A_-}{2} \cos \phi_- = 0$$

$$\textcircled{iv} \quad \dot{x}_2(0) = -\frac{A_+}{2} \omega \sin \phi_+ + \frac{A_-}{2} \sqrt{3} \omega \sin \phi_- = 0$$

ADD $\textcircled{ii} + \textcircled{iv}$
$$-A_+ \omega \sin \phi_+ = 0$$

SUBTRACT $\textcircled{ii} - \textcircled{iv}$
$$-A_- \sqrt{3} \omega \sin \phi_- = 0$$

$$\boxed{\phi_+ = \phi_- = 0}$$

Then \textcircled{i}
$$\frac{A_+}{2} + \frac{A_-}{2} = 0.3 \text{ m} ; \quad \frac{A_+}{2} - \frac{A_-}{2} = 0$$

$$\boxed{A_+ = A_- = 0.3 \text{ m}}$$

< OVERHEAD : PLOTS >