Physics 3210, Spring 2019 Homework #9

Due *in class* Thursday April 11^{th}

Central Force Motion:

1. *K&K* Problem 10.2, *except* do it in MKS units with $m = 0.075$ kg, force $= 80\ r^3$ Newtons and $L = 2.0 \times 10^{-4}\ \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$.
2. *K&K* Problem 10.4.
3. *K&K* Problem 10.6.
4. *K&K* Problem 10.9.

1) K&K 10.2

A PARTICLE OF MASS $m = 0.075$ kg MOVES UNDER AN ATTRACTIVE CENTRAL FORCE OF MAGNITUDE $80\,r^3$ NEWTONS. THE ANGULAR MOMENTUM IS EQUAL TO 2×10^{-4} kg m²/s.

a) $$U_{eff} = \frac{L^2}{2mr^2} + U(r)$$

$$-\frac{dU(r)}{dr} = F(r)\hat{r} = -\left(\frac{80\,N}{m^3}\right)\cdot r^3 = -Ar^3$$

SO $$U(r) = \frac{20\,N}{m^3}\,r^4$$

AND $$\frac{L^2}{2m} = \frac{(2\times10^{-4}\ \text{kg m}^2/\text{s})^2}{2\cdot 0.075\ \text{kg}} = 2.67\times10^{-7}\ \text{Joule}\cdot m^2$$

SO $$\boxed{U_{eff} = \frac{2.67\times10^{-7}\ \text{Joule}\cdot m^2}{r^2} + 20\,\frac{\text{Joule}}{m^4}\cdot r^4}$$

b) CIRCULAR MOTION OCCURS AT THE BOTTOM of U_{eff} CURVE

$$U_{eff} = \frac{L^2}{2mr^2} + \frac{1}{4}Ar^4$$

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_{circ}} = 0 = \frac{-L^2}{mr_{circ}^3} + Ar_{circ}^3$$

$$r_{circ} = \left(\frac{L^2}{mA}\right)^{1/6} = \left(\frac{2\cdot 2.67\times10^{-7}\ J\cdot m^2}{4\cdot 20\ J/m^4}\right)^{1/6}$$

$$\boxed{r_{circ} = 0.04338\ m}$$

AND $$\boxed{U_{eff}(r_{circ}) = 0.02125\ mJ}$$

< SEE ATTACHED FIGURE >

↳ THIS IS TOTAL energy FOR A CIRCULAR ORBIT.

⟹

12) $$U_{eff}(r_0) = U_{eff}(2r_0)$$

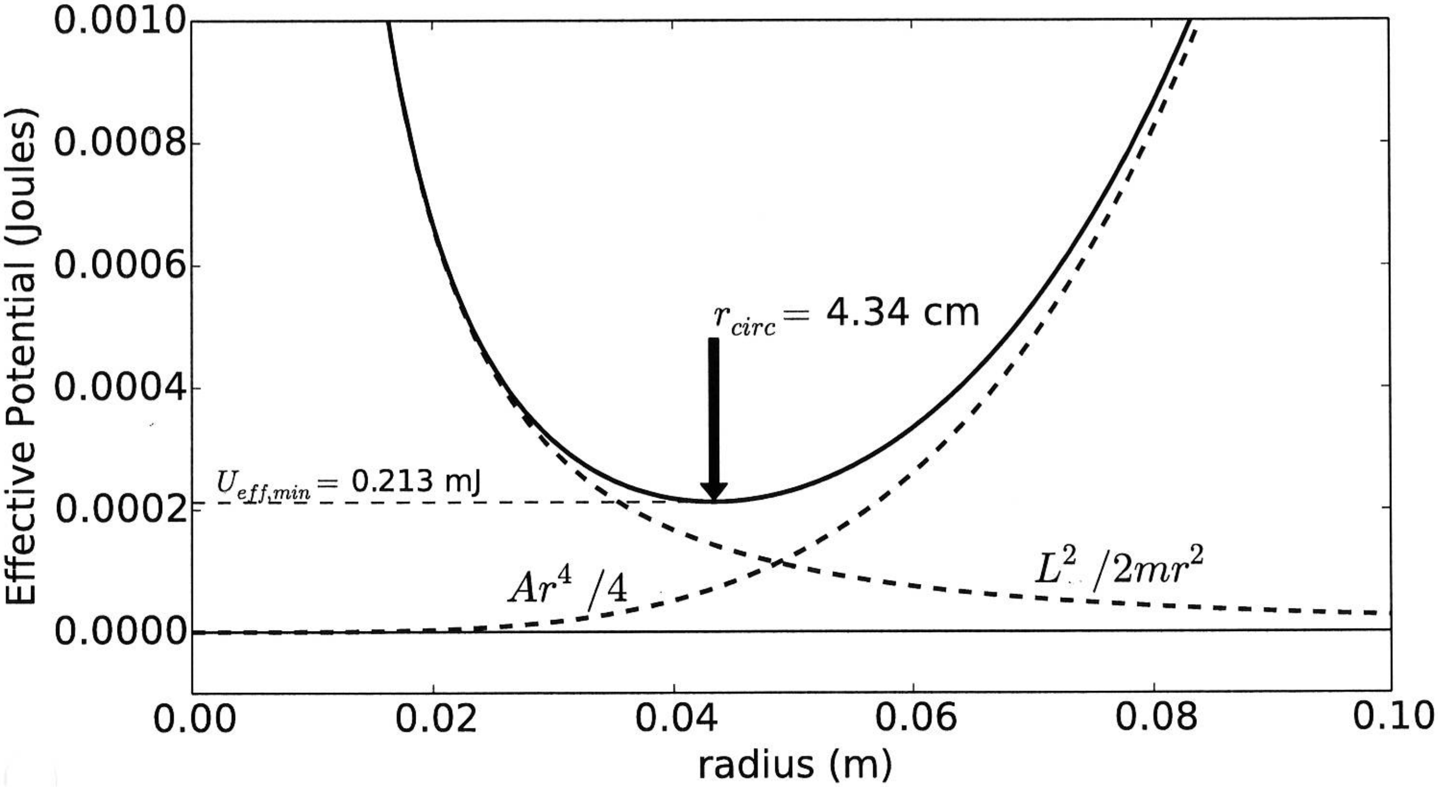
$$\frac{L^2}{2mr_0^2} + \frac{1}{4}Ar_0^4 = \frac{L^2}{2m(2r_0)^2} + \frac{1}{4}A(2r_0)^4 = \frac{L^2}{8mr_0^2} + 4Ar_0^4$$

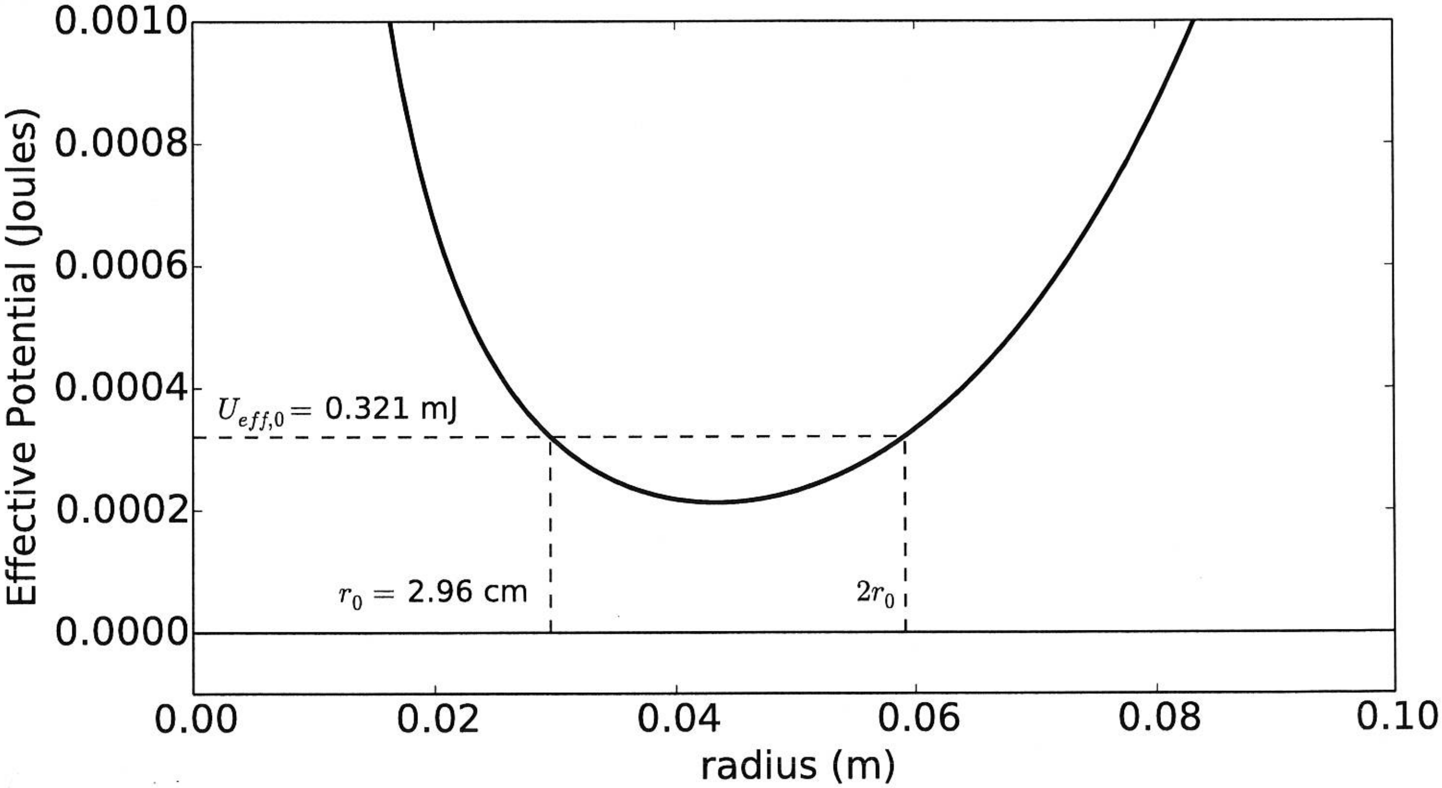
$$\frac{L^2}{2m} + \frac{1}{4}Ar_0^6 = \frac{L^2}{8m} + 4Ar_0^6$$

$$\frac{3}{8}\frac{L^2}{m} = \frac{15}{4}Ar_0^6$$

$$r_0 = \left(\frac{1}{10}\frac{L^2}{mA}\right)^{1/6} = \boxed{2.96\text{ cm}}$$

<SEE ATTACHED FIGURE>


0.0010
0.0008
0.0006
0.0004
0.0002
0.0000
Effective Potential (Joules)
$r_{circ} = 4.34$ cm
$U_{eff,min} = 0.213$ mJ
$Ar^4/4$
$L^2/2mr^2$
0.00
0.02
0.04
0.06
0.08
0.10
radius (m)


0.0010
0.0008
0.0006
0.0004
0.0002
0.0000
Effective Potential (Joules)
$U_{eff,0} = 0.321$ mJ
$r_0 = 2.96$ cm
$2r_0$
0.00
0.02
0.04
0.06
0.08
0.10
radius (m)

2) K&K 10.4

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{A}{r^n}$$

THE CONDITION FOR A CIRCULAR ORBIT TO EXIST IS THAT THERE BE A MINIMUM TO U_{eff} AT SOME finite radius r_0:

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = \frac{-L^2}{mr_0^3} + \frac{nA}{r_0^{n+1}} = 0$$

$$\frac{nA}{r_0^{n+1}} = \frac{L^2}{mr_0^3} \qquad (i)$$

$$\left.\frac{d^2U_{eff}}{dr^2}\right|_{r=r_0} = \frac{3L^2}{mr_0^4} - \frac{n(n+1)A}{r_0^{n+2}} > 0 \qquad (ii)$$

SUBSTITUTING (i) into (ii)

$$\frac{3L^2}{mr_0^4} - \frac{(n+1)L^2}{mr_0^4} > 0$$

$$3 > n+1$$

or $\boxed{n < 2}$

One problematic case would be $n=0$, which we can see from (i) would imply an infinite radius.

$\hookrightarrow$ $\boxed{n<2, \text{ excluding } n=0 \text{ has CIRCULAR ORBIT}}$

3) K & K 10.6

$$\vec{F} = -Kr^4\hat{r} = \frac{-dU}{dr}, \quad U = \frac{1}{5}Kr^5$$

$$U_{eff} = \frac{L^2}{2mr^2} + \frac{1}{5}Kr^5$$

Circular motion will occur at minimum point on U_{eff} curve:

$$\left.\frac{dU_{eff}}{dr}\right|_{r=r_0} = \frac{-L^2}{mr_0^3} + Kr_0^4 = 0$$

$$r_0 = \left(\frac{L^2}{Km}\right)^{1/7}$$

$$E = U_{eff}(r_0) = \frac{L^2}{2m}\left(\frac{Km}{L^2}\right)^{2/7} + \frac{1}{5}K\left(\frac{L^2}{Km}\right)^{5/7}$$

$$= \boxed{\frac{7}{10}\left(\frac{L^{10}K^2}{m^5}\right)^{1/7}}$$

Frequency of small oscillations: Perform Taylor Series Expansion of U_{eff} about $r = r_0$ . . .

$$U_{eff}(r) = U_{eff}(r_0) + \underbrace{\cancel{\frac{dU_{eff}(r_0)}{dr}}}_{=0}\cdot r + \frac{1}{2}\underbrace{\frac{d^2U_{eff}(r_0)}{dr}}_{\text{"effective spring constant" } k}\cdot r^2 + \cdots$$

$$k = \left.\frac{d^2U_{eff}}{dr}\right|_{r=r_0} = \left(\frac{3L^2}{mr_0^4} + 4Kr_0^5\right) = 7K\left(\frac{L^2}{Km}\right)^{3/7} \quad \text{(ARITHMETIC SKIPPED)}$$

$$\boxed{\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{7K\left(\frac{L^2}{Km}\right)^{3/7}}{m}}}$$

4) K & K 10.9 Halley's Comet

a) Determine perhelion & aphelion distances (r_{min} & r_{max} in figure pg 391).

$$r_{per} = \frac{r_0}{1+\epsilon}$$

$$r_{ap} = \frac{r_0}{1-\epsilon}$$

Eq 10.28 $$A = r_{per} + r_{ap} = r_0\left(\frac{1}{1+\epsilon} + \frac{1}{1-\epsilon}\right) = \frac{2 r_0}{1-\epsilon^2}$$

Eq 10.31 $$T^2 = \frac{\pi^2 m}{2C} A^3$$

$$\therefore \quad A = \left(\frac{2T^2 G m_{sun}}{\pi^2}\right)^{1/3} = 5.37 \times 10^{12} \text{ meters}$$

and

$$r_0 = \frac{1}{2} A\left(1-\epsilon^2\right) = 1.74 \times 10^{11} \text{ m}$$

$$\therefore \quad \boxed{\begin{aligned} r_{per} &= \frac{r_0}{1+\epsilon} = 8.86 \times 10^{10} \text{ m} \\ r_{ap} &= \frac{r_0}{1-\epsilon} = 5.27 \times 10^{12} \text{ m} \end{aligned}}$$

b) What is speed at perhelion?

Eq 10.19 $$r_0 = \frac{L^2}{m G m_{sun} m} = \frac{\cancel{m^2} v_{per}^2 r_{per}^2}{\cancel{m} G m_{sun} \cancel{m}}$$

$$V_{per} = \left(\frac{G m_{sun} r_0}{r_{per}^2}\right)^{1/2} = \boxed{5.42 \times 10^4 \text{ m/s}}$$