Physics 3210, Spring 2019 — Homework #6

Due *in class* Thursday February 28^{th}

Lagrangian Mechanics:

1. Two blocks, both of mass M, are connected with a spring with spring constant k. They slide in the x-direction on a frictionless surface.

   (a) Use the Lagrangian formalism to find coupled differential equations for the positions x_1 and x_2 of the two blocks.

   (b) Use these equations to show that the center-of-mass of the system moves with constant velocity.

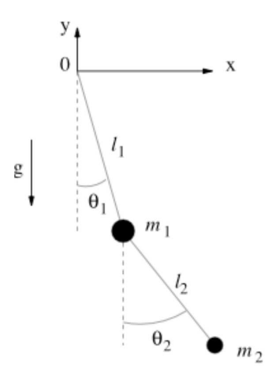
   (c) Find the period of oscillation of the two masses with respect to each other.

2. Consider the *double pendulum* in the figure below, consisting of two bobs of mass m_1 and m_2 at the ends of massless rods of length l_1 and l_2 in which the second rod swings from the first mass.

   (a) Derive (but do not solve) the equations of motion for the two bobs in terms of their coordinates θ_1 and θ_2. You may find the arithmetic easier if you make use of the trigonometric identity

   $$\cos A \cos B + \sin A \sin B = \cos(A - B)$$

   (b) Find the simplified version of these equations for the case in which θ_1 and θ_2 are both small angles.

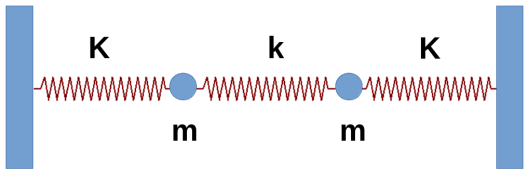


Additional problems on the back of this page.

Normal Modes of Oscillation:

3. *K&K* Problem 6.3.

4. See the figure below. Three springs and two equal masses lie between two walls. The spring constant K of the outer springs is larger than the spring constant k of the inner spring. Let x_1 and x_2 be the positions of the left and right masses, respectively, relative to their equilibrium positions.



   (a) Find general oscillatory solutions for $x_1(t)$ and $x_2(t)$.

   (b) Assume $K/m = 1.0/\mathrm{s}^2$ and $k = 0.2\,K$. Plot x_1 and x_2 versus time for the normal modes of the system. Assume the equilibrium distance between the two masses is 1.0 m.

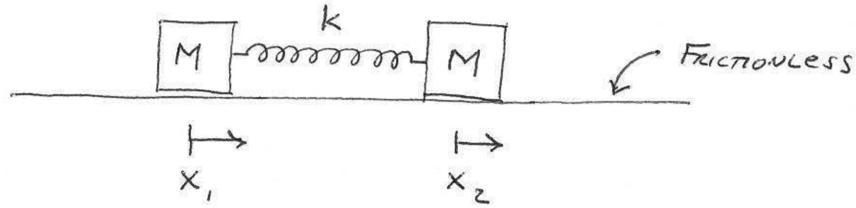
   (c) Now, find the particular solutions for x_1 and x_2, for the initial conditions:

$$\begin{aligned} x_1(0) &= 0.0 \text{ m} \\ \dot{x}_1(0) &= 0.0 \text{ m/s} \\ x_2(0) &= 0.4 \text{ m} \\ \dot{x}_2(0) &= 0.0 \text{ m/s} \end{aligned}$$

   Plot these solutions for t in the range $0 - 100$ s.

   (Note: If you don't have access to a graphing program on your computer, have a look at online plotting tools such as **WolframAlpha** and **fooplot.com**.)

1)



a) $L = K - U = \frac{1}{2}M\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 - \frac{1}{2}k(x_2 - x_1)^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = M\ddot{x}_1 - k(x_2 - x_1) = 0$$

$$① \quad \boxed{\ddot{x}_1 - \omega^2 x_2 + \omega^2 x_1 = 0} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = M\ddot{x}_2 + k(x_2 - x_1) = 0$$

$$② \quad \boxed{\ddot{x}_2 + \omega^2 x_2 - \omega^2 x_1 = 0}$$

b) ADD EQNS ① AND ②:

$$\ddot{x}_1 + \ddot{x}_2 = 0$$

ACCELERATION OF C.O.M. $= \dfrac{M\ddot{x}_1 + M\ddot{x}_2}{2M} = \dfrac{1}{2}\left(\ddot{x}_1 + \ddot{x}_2\right) = 0$

$\therefore$ COM MOVES w/ CONSTANT $\vec{V}$

c) SUBTRACT EQN ① − ②

$$(\ddot{x}_1 - \ddot{x}_2) + 2\omega^2(x_1 - x_2) = 0$$

SIMPLE HARMONIC OSCILLATOR IN $(x_1 - x_2)$ w/ FREQ $\omega' = \sqrt{2}\,\omega = \sqrt{\frac{2k}{m}}$

PERIOD OF OSCILLATION $\boxed{T = \dfrac{2\pi}{\omega'} = \pi\sqrt{\dfrac{2m}{k}}}$

2) EASIEST TO START BY WRITING DOWN LAGRANGIAN IN CARTESIAN COORDS, THEN USE TRIGONOMETRY TO CONVERT:

a) $L = K - U = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - m_1 g y_1 - m_2 g y_2$

Now,

$$x_1 = l_1 \sin\theta_1 \qquad \dot{x}_1 = l_1 \dot{\theta}_1 \cos\theta_1$$

$$y_1 = -l_1 \cos\theta_1 \qquad \dot{y}_1 = l_1 \dot{\theta}_1 \sin\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \qquad \dot{x}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2 \qquad \dot{y}_2 = l_1 \dot{\theta}_1 \sin\theta_1 + l_2 \dot{\theta}_2 \sin\theta_2$$

PLUG THESE INTO EXPRESSION FOR LAGRANGIAN

$$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$
$$+ m_1 g l_1 \cos\theta_1 + m_2 g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

use $\cos A \cos B + \sin A \sin B = \cos(A-B)$

☆ $$L = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ m_1 g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

Plug L into Euler-Lagrange Equations, for generalized coordinates θ_1 & θ_2:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta} = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right]$$
$$+ (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right]$$
$$+ m_2 g l_2 \sin\theta_2 = 0$$

2a) ... CONTINUED

rewriting ...

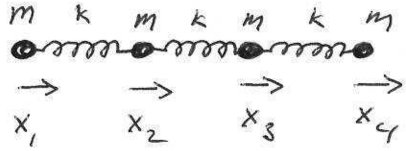
$$\boxed{\begin{aligned} &\ddot{\theta}_1 + \frac{m_2}{m_1+m_2}\frac{l_2}{l_1}\left(\ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2)\right) + \frac{g}{l_1}\sin\theta_1 = 0 \\ &\ddot{\theta}_2 + \frac{l_1}{l_2}\left(\ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_1^2\sin(\theta_1-\theta_2)\right) + \frac{g}{l_2}\sin\theta_2 = 0 \end{aligned}}$$

b) Small angle approximation, $\sin\theta \approx \theta$

$\cos\theta \approx \underline{1}$

$$\boxed{\begin{aligned} &\ddot{\theta}_1 + \left(\frac{m_2}{m_1+m_2}\right)\frac{l_2}{l_1}\left(\ddot{\theta}_2 + \dot{\theta}_2^2(\theta_1-\theta_2)\right) + \frac{g}{l_1}\theta_1 = 0 \\ &\ddot{\theta}_2 + \frac{l_1}{l_2}\left(\ddot{\theta}_1 - \dot{\theta}_1^2(\theta_1-\theta_2)\right) + \frac{g}{l_2}\theta_2 = 0 \end{aligned}}$$

3) K&K 6.3



$$L = K - U = \tfrac{1}{2} m\dot{x}_1^2 + \tfrac{1}{2} m\dot{x}_2^2 + \tfrac{1}{2} m\dot{x}_3^2 + \tfrac{1}{2} m\dot{x}_4^2$$
$$- \tfrac{1}{2} k (x_2 - x_1)^2 - \tfrac{1}{2} k (x_3 - x_2)^2 - \tfrac{1}{2} k (x_4 - x_3)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = m\ddot{x}_1 - k(x_2 - x_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = m\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_3}\right) - \frac{\partial L}{\partial x_3} = m\ddot{x}_3 + k(x_3 - x_2) - k(x_4 - x_3) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_4}\right) - \frac{\partial L}{\partial x_4} = m\ddot{x}_4 + k(x_4 - x_3) = 0$$

LET $\omega_0 = \sqrt{\frac{k}{m}}$, $\beta = \frac{\omega^2}{\omega_0^2}$, $\ddot{x}_i = -\omega^2 x_i$

THEN ABOVE CAN BE WRITTEN AS

$$\beta x_1 = (x_1 - x_2)$$
$$\beta x_2 = (2x_2 - x_1 - x_3)$$
$$\beta x_3 = (2x_3 - x_2 - x_4)$$
$$\beta x_4 = (x_4 - x_3)$$

BY SYMMETRIES & CONDITION THAT C.O.M. IS AT REST, POSSIBLE NORMAL MODES ARE

I $(x_4 = x_1)$ AND $(x_3 = x_2)$

II $(x_4 = -x_1)$ AND $(x_3 = -x_2)$

$\longrightarrow$

CASE I UNIQUE EQNS ARE

(i) $\beta x_1 = x_1 - x_2$

(ii) $\beta x_2 = (2x_2 - x_1 - x_2) = (x_2 - x_1)$

$$\underbrace{x_1(1-\beta) = x_2}_{(i)} \underbrace{= \frac{x_1}{(1-\beta)}}_{(ii)}$$

$$1+\beta^2 - 2\beta = 1$$

$$\beta^2 = 2\beta$$

$$\beta = 2 = \frac{\omega^2}{\omega_0^2} \; ; \quad \boxed{\omega = \sqrt{\frac{2k}{m}}}$$

CASE II UNIQUE EQUATIONS ARE

(iii) $\beta x_1 = x_1 - x_2$

(iv) $\beta x_2 = 2x_2 - x_1 + x_2 = 3x_2 - x_1$

$$\underbrace{x_1(1-\beta) = x_2}_{(iii)} \underbrace{= \frac{x_1}{3-\beta}}_{(iv)}$$

$$(1-\beta)(3-\beta) = 1$$

$$\beta^2 - 4\beta + 2 = 0$$

$$\beta = 2 \pm \sqrt{2} = \frac{\omega^2}{\omega_0^2}$$

$$\therefore \boxed{\omega = \sqrt{\frac{(2 \pm \sqrt{2})k}{m}}}$$

4) $L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} K x_1^2 - \frac{1}{2} k (x_2 - x_1)^2 - \frac{1}{2} K x_2^2$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = m\ddot{x}_1 + K x_1 - k(x_2 - x_1) = 0$$

$$\ddot{x}_1 + \left(\frac{k+K}{m}\right) x_1 - \frac{k}{m} x_2 = 0 \qquad (i)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = m\ddot{x}_2 + K x_2 + k(x_2 - x_1) = 0$$

$$\ddot{x}_2 + \left(\frac{k+K}{m}\right) x_2 - \frac{k}{m} x_1 = 0 \qquad (ii)$$

ADD (i) & (ii) $\quad (\ddot{x}_1 + \ddot{x}_2) + \frac{K}{m}(x_1 + x_2) = 0$

SUBTRACT (i) - (ii) $\quad (\ddot{x}_1 - \ddot{x}_2) + \left(\frac{2k+K}{m}\right)(x_1 - x_2) = 0$

So $x_+ = x_1 + x_2$ AND $x_- = x_1 - x_2$ HAVE GENERAL SOLUTIONS

$$x_+ = A_+ \cos(\omega_+ t + \phi_+) \qquad \omega_+ = \sqrt{\frac{K}{m}}$$

$$x_- = A_- \cos(\omega_- t + \phi_-) \qquad \omega_- = \sqrt{\frac{2k+K}{m}}$$

AND $\quad x_1 = \frac{x_+ + x_-}{2} = \frac{A_+}{2}\cos(\omega_+ t + \phi_+) + \frac{A_-}{2}\cos(\omega_- t + \phi_-)$

$$x_2 = \frac{x_+ - x_-}{2} = \frac{A_+}{2}\cos(\omega_+ t + \phi_+) - \frac{A_-}{2}\cos(\omega_- t + \phi_-)$$

APPLY INITIAL CONDITIONS $x_1(t=0) = 0$, $x_2(t=0) = 0.4\,m$, $\dot{x}_1(0) = \dot{x}_2(0) = 0$

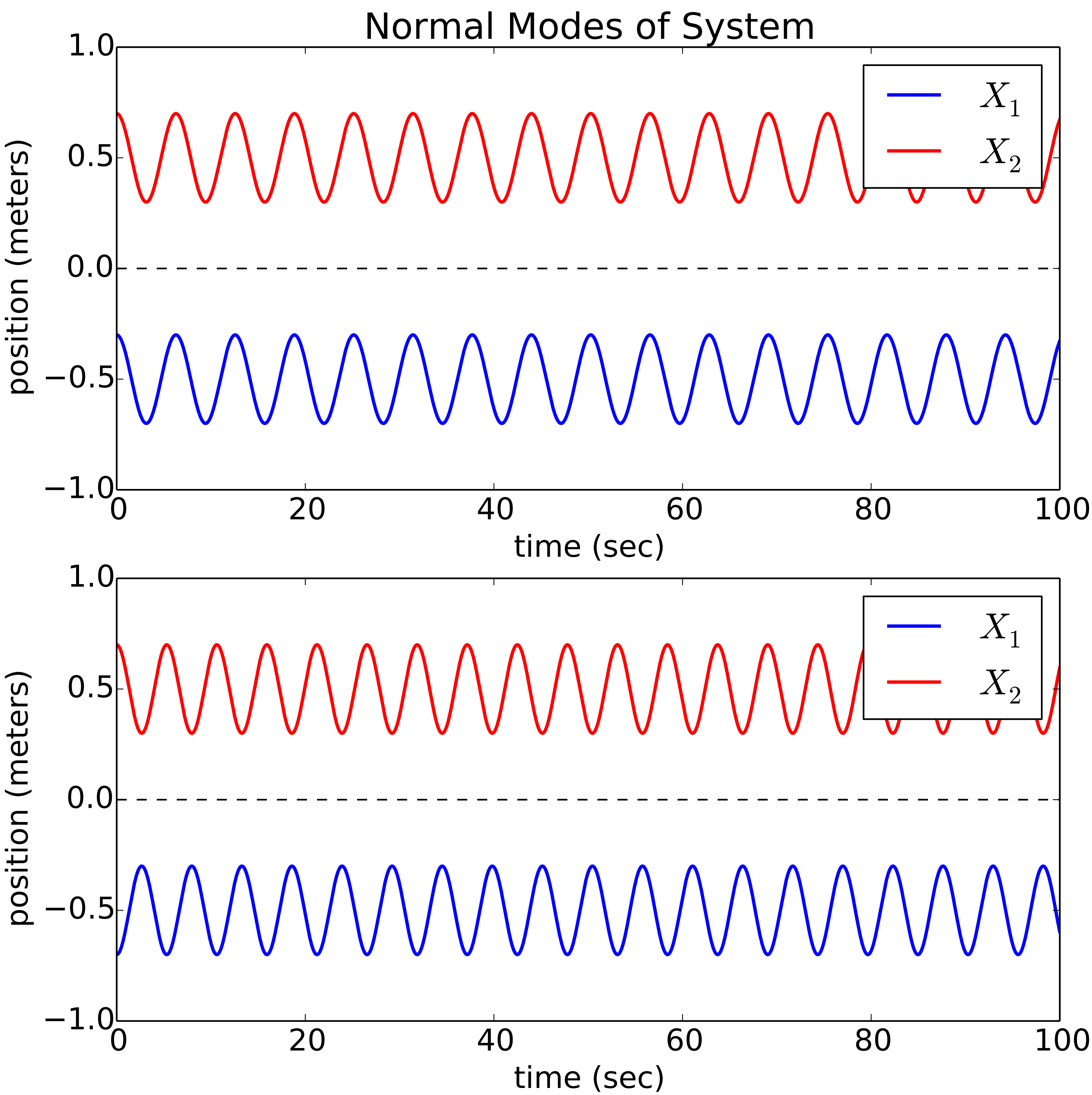
$\Rightarrow$ VELOCITY IC's give $\phi_+ = \phi_- = 0$

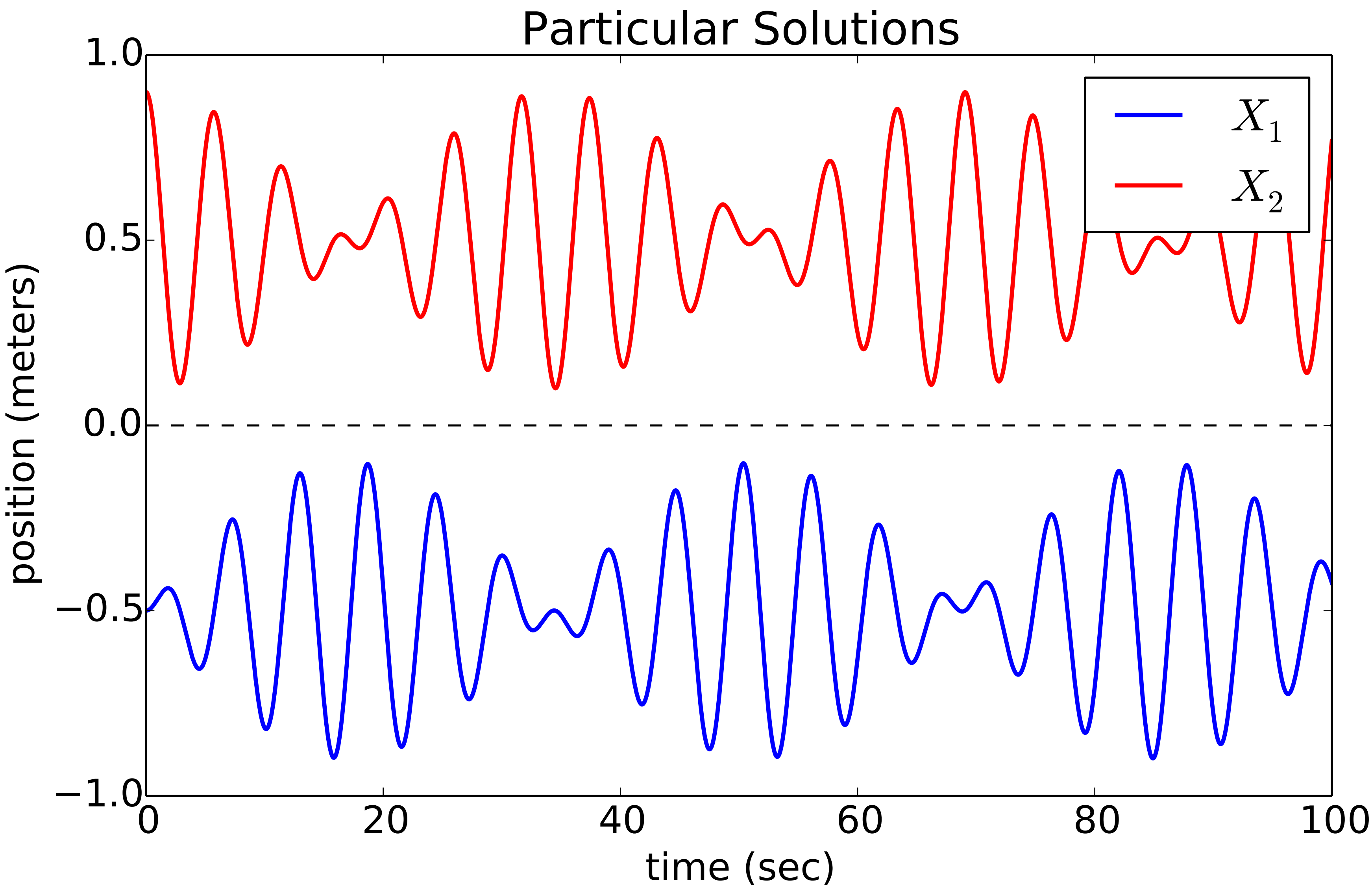
$\Rightarrow$ THEN POSITION IC's $\quad \frac{A_+}{2} + \frac{A_-}{2} = 0$

$$\frac{A_+}{2} - \frac{A_-}{2} = 0.4$$

$$\hookrightarrow \boxed{A_+ = 0.4\,m, \; A_- = -0.4\,m}$$

See Attached Plots $\longrightarrow$


Normal Modes of System
position (meters)
time (sec)
X_1
X_2
1.0
0.5
0.0
−0.5
−1.0
0
20
40
60
80
100
position (meters)
time (sec)
X_1
X_2
1.0
0.5
0.0
−0.5
−1.0
0
20
40
60
80
100



Particular Solutions
1.0
0.5
0.0
−0.5
−1.0
position (meters)
0
20
40
60
80
100
time (sec)
X_1
X_2