Physics 3210, Spring 2019 Homework #4

Due *in class* Thursday February 14^{th}

Gravity and Harmonic Oscillator:

1. *K&K* Problem 3.15.

Center of Mass:

2. *K&K* Problem 4.1.
3. *K&K* Problem 4.2.

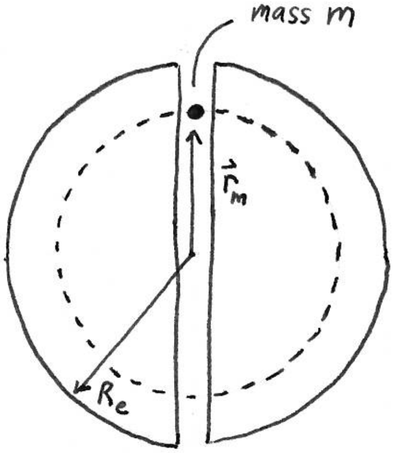
Momentum Conservation:

4. *K&K* Problem 4.5.
5. *K&K* Problem 4.7. *Note that there are two cases to be considered: (a) when m_1 is in contact with the wall, and (b) when m_1 loses contact with the wall.*

Rocket Motion:

6. A rocket that starts with mass M ejects exhaust at a given speed u. What is the mass of the rocket (including unused fuel) when its momentum is maximum?

1) K & K 3.15



THE FORCE ON MASS m IS ONLY DUE TO THAT FRACTION OF THE EARTH'S MASS AT $r < r_m$:

$$\vec F = -\frac{G M(r<r_m)\, m}{r_m^2}\,\hat r_m$$

NOW

$$M(r<r_m) = \rho \cdot \frac{4}{3}\pi r_m^3$$

WHERE

$$\rho = \text{density} = \frac{M_e}{\frac{4}{3}\pi R_e^3}$$

So,

$$\vec F = -\left(\frac{G M_e m}{R^3}\right) r_m\, \hat r_m$$

THAT IS, m EXPERIENCES A SPRINGLIKE LINEAR RESTORING FORCE WITH

$$k \to \frac{G M_e m}{R_e^3}$$

AND WILL OBEY SIMPLE HARMONIC OSCILLATOR EQUATION

$$\frac{d^2 r_m}{dt^2} + \frac{k}{m} r_m = \frac{d^2 r_m}{dt^2} + \frac{G M_e}{R^3} r_m = \frac{d^2 r_m}{dt^2} + \omega^2 r_m = 0$$

THE "TIME TO RETURN TO POINT OF DEPARTURE" = PERIOD $T = \frac{2\pi}{\omega}$

$$\boxed{T = 2\pi\sqrt{\frac{R^3}{G M_e}}} \quad (= 5{,}061 \text{ sec} = 1 \text{ hr } 24 \text{ min } 21 \text{ sec})$$

$\Rightarrow$ A SATELLITE CIRCLING EARTH WILL ACCELERATE AT $g = \frac{G M_e}{R_e^2} = \frac{V^2}{R_e}$

$$V = \frac{(\text{circumference})}{(\text{period})} = \frac{2\pi R_e}{T}$$

& SO

$$\frac{G M_e}{R_e^2} = \left(\frac{2\pi R_e}{T}\right)^2 \cdot \frac{1}{R_e} \to T = 2\pi\sqrt{\frac{R_e^3}{G M_e}}; \text{ Same!}$$

2) K & K 4.1

Rod has mass/length $\lambda(x) = A\cos\left(\frac{\pi x}{2L}\right) \quad 0 < x < L$

a) Total mass of rod

$$M = \int dm = \int_{x=0}^{L} \lambda(x)\,dx = \int_{x=0}^{L} A\cos\left(\frac{\pi x}{2L}\right)dx$$

$$= \frac{2LA}{\pi}\sin\left(\frac{\pi x}{2L}\right)\Bigg|_{x=0}^{L} = \frac{2LA}{\pi}$$

$$\boxed{M = \frac{2LA}{\pi}}$$

b) $$x_{cm} = \frac{1}{M}\int \lambda(x)\,x\,dx = \frac{\pi}{2LA}\int_{x=0}^{L} A\cos\left(\frac{\pi x}{2L}\right)x\,dx$$

Now $$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

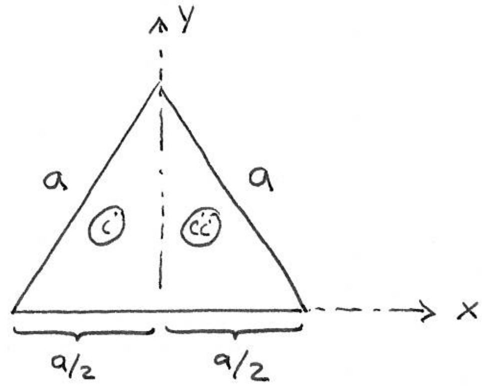
So $$x_{cm} = \frac{\pi}{2L}\cdot\left[\left(\frac{2L}{\pi}\right)^2\cos\left(\frac{\pi x}{2L}\right) + \left(\frac{2L}{\pi}\right)x\sin\left(\frac{\pi x}{2L}\right)\right]_0^L$$

$$= \frac{\pi}{2L}\left[\left(\frac{2L}{\pi}\right)^2\cdot(0-1) + \left(\frac{2L}{\pi}\right)(L\cdot 1 - 0\cdot 0)\right]$$

$$= \frac{\pi}{2L}\left[\frac{2L^2}{\pi} - \frac{4L^2}{\pi^2}\right]$$

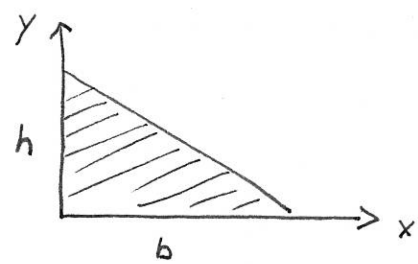
$$\boxed{x_{cm} = L\cdot\left(1-\frac{2}{\pi}\right)}$$

3) K & K 4.2



FIND CENTER-OF-MASS

WE HAVE THE IN-CLASS RESULT THAT



$$\vec{R}_{cm} = \frac{b}{3}\hat{x} + \frac{h}{3}\hat{y}$$

APPLY THIS TO SIDES (i) & (ii) of EQUILATERAL TRIANGLE ...

$$\vec{R}_{cm,i} = \left(\frac{-a/2}{3}\right)\hat{x} + \left(\frac{\sqrt{a^2-(a/2)^2}}{3}\right)\hat{y}$$

$$\vec{R}_{cm,ii} = \left(\frac{+a/2}{3}\right)\hat{x} + \left(\frac{\sqrt{a^2-(a/2)^2}}{3}\right)\hat{y}$$

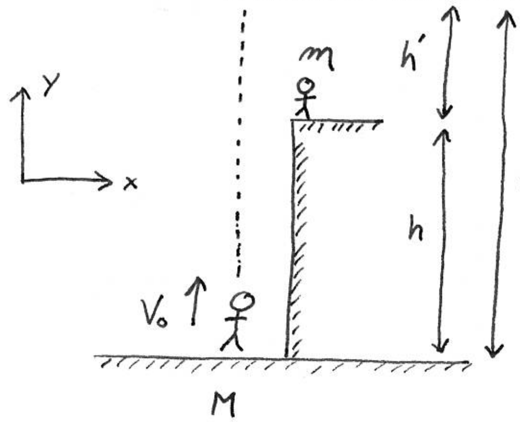
$$\vec{R}_{EQUILATERAL} = \text{Average of } \vec{R}_{cm,i} \text{ and } \vec{R}_{cm,ii}$$

$$= \frac{\vec{R}_{cm,i} + \vec{R}_{cm,ii}}{2}$$

$$= 0\hat{x} + a\cdot\frac{\sqrt{1-1/4}}{3}\hat{y} = \boxed{\frac{a}{2}\sqrt{\frac{1}{3}}\,\hat{y}}$$

$\Rightarrow$ THE CENTER-OF-MASS IS LOCATED $\frac{a}{2\sqrt{3}}$ ABOVE CENTER OF BASE OF EQUILATERAL TRIANGLE w/ SIDE a.

4) K & K 4.5



First, we need to relate y, v, a for uniform acceleration without reference to time:

$$y = y_0 + v_0 t + \frac{1}{2} a t^2 \quad (i)$$

$$v = v_0 + at, \quad t = \frac{v - v_0}{a} \quad (ii)$$

Substitute (ii) into (i) to obtain $v^2 = v_0^2 + 2a(y - y_0)$ (iii)

1) Projectile motion until just before acrobat grabs monkey at h:

using (iii) $\quad v_{hi}^2 = v_0^2 - 2gh$

2) Momentum conservation as acrobat grabs monkey

$$M v_{hi} = (M + m) V_{hf}$$

3) Projectile motion after acrobat grabs monkey

using (iii) $\quad V_H^2 = 0 = V_{hf}^2 - 2gh'$

Combine...

$$h' = \frac{V_{hf}^2}{2g} = \frac{1}{2g}\left(\frac{M}{m+M}\right)^2 V_{hi}^2 = \frac{1}{2g}\left(\frac{M}{m+M}\right)^2 \left(V_0^2 - 2gh\right)$$

Final height $\boxed{H = h + h' = \left(\frac{M}{m+M}\right)^2 \frac{V_0^2}{2g} + \left[1 - \left(\frac{M}{m+M}\right)^2\right] h}$

5) K & K 4.7

THERE ARE TWO SITUATIONS WHICH NEED TO BE DESCRIBED: (1) WHILE M_1 IS IN CONTACT WITH WALL AND AN EXTERNAL FORCE (NORMAL FORCE OF WALL ON M_1) INFLUENCES SYSTEM, AND (2) AFTER M_1 LOSES CONTACT WITH WALL AND SYSTEM IS ISOLATED.

1) WHILE M_1 IS IN CONTACT WITH WALL, M_2 MOVES AS A SIMPLE OSCILLATOR WITH $\underbrace{\omega = \sqrt{\frac{k}{m_2}}}$

$$X_1 = 0$$

$$X_2 = \left(1 - \tfrac{1}{2}\cos\omega t\right)\ell \qquad \dot{X}_2 = \frac{\omega \ell}{2}\sin\omega t$$

$$\boxed{X_{cm}\left(t < \tfrac{\pi}{2\omega}\right) = \frac{M_1 X_1 + M_2 X_2}{M_1 + M_2} = \left(\frac{m_2}{m_1 + m_2}\right)\left(1 - \tfrac{1}{2}\cos\omega t\right)\ell}$$

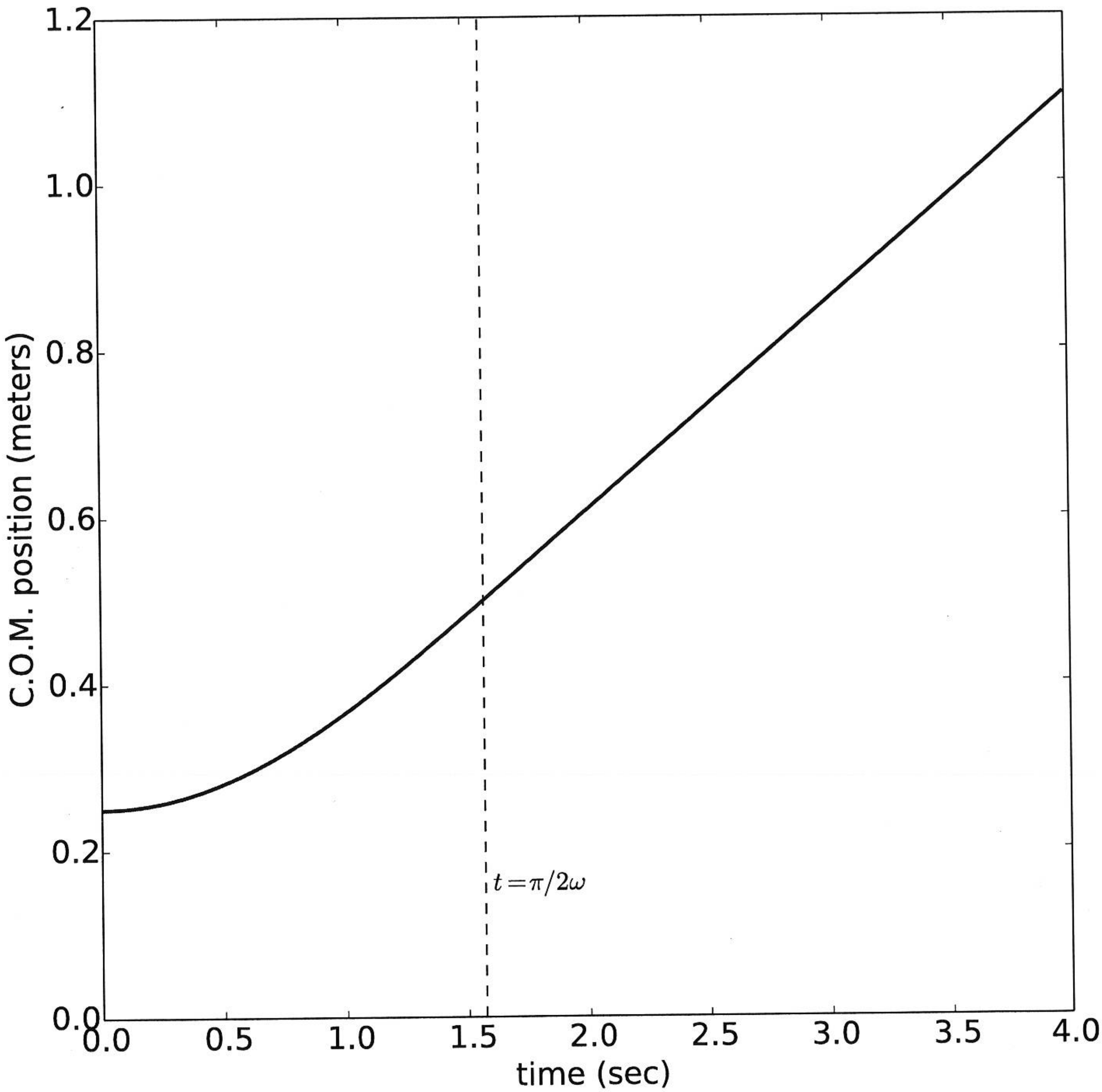
2) WHEN $t = \frac{\pi}{2\omega}$, M_2 WILL BE AT EQUILIBRIUM POSN $X_2 = \ell$. AT THIS POINT, THE NORMAL FORCE ON $M_1 = 0$ AND THE C-M OF THE ISOLATED SYSTEM MOVES AT CONSTANT VELOCITY.
THIS WILL JUST BE THE INSTANTANEOUS VELOCITY OF THE CM @ $t = \pi/2\omega$

$$\dot{X}_2\left(t = \pi/2\omega\right) = \frac{\omega \ell}{2}\sin\left(\omega \cdot \pi/2\omega\right) = \frac{\omega \ell}{2}$$

SO

$$\boxed{X_{cm}\left(t > \tfrac{\pi}{2\omega}\right) = \left(\frac{m_2}{m_1 + m_2}\right)\left[\frac{\omega \ell}{2}\left(t - \frac{\pi}{2\omega}\right) + \ell\right]}$$

$\Rightarrow$ SEE ATTACHED GRAPH OF X_{cm} VS t FOR $m_1 = m_2 = k = \ell = 1$.



C.O.M. position (meters)
time (sec)
$t=\pi/2\omega$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

6) ROCKET MOTION

See Ex 4.16 IN TEXT. RESULT IS

$$V_f - V_0 = -u \ln\left(\frac{M_0}{M_f}\right)$$

IN 1-D. Recall u is in direction opposite V, i.e. if V is positive u is negative.

IF ROCKET STARTS AT REST, $V_0 = 0$. THEN

MOMENTUM $p = MV = -Mu \ln\left(\frac{M_0}{M}\right)$

MAXIMIZE p:

$$\frac{dp}{dm} = -u \ln\left(\frac{M_0}{M}\right) + u = 0$$

$$\ln\left(\frac{M_0}{M}\right) = 1$$

$$M = \frac{M_0}{e}$$

IS MASS WHEN MOMENTUM IS MAXIMUM

$$p_{max} = -\frac{M_0}{e} u \ln\left(\frac{M_0}{M_0/e}\right)$$

$$\boxed{p_{max} = \frac{-m_0 u}{e}}$$