Physics 3210 Spring 2019 Discussion #22 Answers

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April 10th 2019

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We have seen that for the case of a critically damped oscillator our usual guess for the solution to the linear second order homogeneous differential equation fails. Now we must make another guess for a possible solution to the critically damped oscillator. If one is astute we may use the results of our first initial guess to make a good next guess. We have seen that with the first initial guess we obtain three possible exponentials, and for the critically damped case we got the following:

$$x(t) = Ae^{-\gamma t} \tag{1}$$

We will now assume that the constant in (1) is not a constant.

$$x(t) = u(t)e^{-\gamma t} \tag{2}$$

The goal now is to see if we can find u(t) such that the guess satisfies the differential equation. We will begin by simply plugging the guess into the differential equation and seeing what properties u(t) must have. To plug it in we need to take the first and second derivatives of (2):

$$x'(t) = u'(t)e^{-\gamma t} - \gamma u(t)e^{-\gamma t}$$
(3)

$$x''(t) = u''(t)e^{-\gamma t} - 2\gamma u'(t)e^{-\gamma t} + \gamma^2 u(t)e^{-\gamma t}$$
(4)

Now we will plug (2), (3), and (4) into the DE:

$$u''(t)e^{-\gamma t} - 2\gamma u'(t)e^{-\gamma t} + \gamma^2 u(t)e^{-\gamma t} + 2\gamma (u'(t)e^{-\gamma t} - \gamma u(t)e^{-\gamma t}) + \omega^2 u(t)e^{-\gamma t}$$

= $e^{-\gamma t}(u''(t) - 2\gamma u'(t) + \gamma^2 u(t) + 2\gamma u'(t) - 2\gamma^2 u(t) + \gamma^2 u(t))$ (5)

We can see here that all but one term cancels out.

$$e^{-\gamma t}(u''(t) - 2\gamma u'(t) + \gamma^2 u(t) + 2\gamma u'(t) - 2\gamma^2 u(t) + \gamma^2 u(t))$$

= $u''(t)e^{-\gamma t} = 0$ (6)

We know that the exponential term won't be zero, thus we find the following property of u(t):

$$u''(t) = 0 \tag{7}$$

We can integrate this expression twice and come to the following general form of u(t):

$$u(t) = At + B \tag{8}$$

Thus, the following general solution satisfies the differential equation:

$$x(t) = (At + B)e^{-\gamma t} \tag{9}$$

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For this problem we are given initial conditions. Let's use those initial conditions to find a particular solution before answering the main question. The first initial condition gives the following:

$$x(0) = B = x_0 \tag{10}$$

The second initial condition tells us the block is given an initial velocity in the negative \hat{x} direction. Taking a derivative of (9) and plugging in the initial condition gives the following:

$$\begin{aligned} x'(0) &= A - \gamma B = -v_0 \\ &\implies A = \gamma x_0 - v_0 \end{aligned} \tag{11}$$

Thus we obtain the following particular solution:

$$x(t) = (x_0 + (\gamma x_0 - v_0)t)e^{-\gamma t}$$
(12)

Now that we have a particular solution we can determine the conditions required for x(t) to never pass the equilibrium position. Looking at (12) we can note that the only part of the expression that can be negative is the following part:

$$(\gamma x_0 - v_0)t \tag{13}$$

Since this value grows with time, no matter what x_0 is eventually the magnitude of (13) will be larger. If (13) is negative that will make (12) negative eventually and thus have the block pass the equilibrium point. This leads straight to the following condition for the block to never pass the equilibrium point:

$$\gamma x_0 - v_0 \ge 0$$
$$\implies v_0 \le \gamma x_0 \tag{14}$$

From (14) it's very apparent that the maximum velocity allowed is the following:

$$v_0 = \gamma x_0 \tag{15}$$