Physics 3210 Spring 2019 Discussion #21 Answers

Josh Peterson

April 5th 2019

1

This first part of this question asks how much energy is required to get the satellite into the initial elliptical orbit. The total energy required is the change in energy of the satellite:

$$\Delta E = E_f - E_i \tag{1}$$

The initial energy is simply the potential energy of the satellite sitting at Cape Canaveral, which happens to have an altitude of approximately zero.

$$E_i = -\frac{GMm_s}{R_E} \tag{2}$$

Where R_E is the radius of the Earth and M is the mass of the Earth. The total energy of an elliptical orbit is given by the following:

$$E_f = -\frac{GMm_s}{r_{max} + r_{min}} \tag{3}$$

We derived this result in discussion and the derivation is also in the text book, so I will not go over the derivation of this result. Plugging (2) and (3) into (1) gives the energy required to get the satellite into the elliptical orbit we want:

$$\begin{aligned}\Delta E &= GMm_E\left(\frac{1}{R_E} - \frac{1}{r_{max} + r_{min}}\right) \\ &= 5.454 \times 10^{10} J \end{aligned} \tag{4}$$

The next part want us to find the eccentricity of the elliptical orbit. For this we will use the following general solution for orbital radius as a function of angle:

$$r(\theta) = \frac{r_0}{1 + \epsilon \cos\theta} \tag{5}$$

We know the maximum and minimum radius for the orbit, so let's obtain those with (5):

$$r_{max} = \frac{r_0}{1-\epsilon} \tag{6}$$

$$r_{min} = \frac{r_0}{1+\epsilon} \tag{7}$$

Now we have two equations with two unknowns with which we can solve for the eccentricity. Let's do this by dividing (6) by (7) and then solve that equation for ϵ. Doing so gives the following eccentricity:

$$\begin{aligned}\epsilon &= \frac{\frac{r_{max}}{r_{min}} - 1}{\frac{r_{max}}{r_{min}} + 1} \\ &= 0.669 \end{aligned} \tag{8}$$

The next part wants us to find the angular momentum of the satellite in the elliptical orbit. To do this we can simply use the definition of the eccentricity, which is the following:

$$\epsilon = \sqrt{1 + \frac{2EL^2}{\mu(GMm_s)^2}} \tag{9}$$

Solving for L, substituting (3), and plugging in known values gives the following:

$$\begin{aligned} L &= \sqrt{\frac{1}{2}m_2^2MG(1-\epsilon^2)(r_{max}+r_{min})} \\ &= 7.113 \times 10^{13} kg\frac{m^2}{s} \end{aligned} \tag{10}$$

The last part of the question asks for the orbital speed of the satellite as it reaches the apogee of the orbit. To do this we will use the definition of angular momentum. It is important to note that the trajectory of the satellite is perpendicular to the position $\overrightarrow{r}$ of the satellite. So:

$$L = \mu r_a v \tag{11}$$

Solving (11) for speed gives the following:

$$\begin{aligned} v &= \frac{L}{m_s r_a} \\ &= 1684\frac{m}{s} \end{aligned}$$

2

For this question it is important to note that the satellite will be in geosynchronous orbit. This means that the angular frequency of the satellite's orbit is the same as the angular frequency of Earth's rotation, ω_E. So we can easily obtain the angular momentum of the satellite in the circular orbit:

$$L = m_s r_a^2 \omega_E \tag{12}$$

We also know that for circular orbits the eccentricity of the orbit is zero. We can then plug that and (11) into (9) and solve for the energy of the satellite in circular orbit:

$$E = -\frac{G^2 M^2 m_s}{2 r_a^4 \omega_E^2} \tag{13}$$

Now the energy required to boost the satellite into circular orbit is the difference in energy of the satellite in the circular orbit and the elliptical orbit. Using (1), (3), and (13) we obtain the following energy:

$$\begin{aligned}\Delta E &= \frac{GMm_s}{r_{max} + r_{min}} - \frac{G^2 M^2 m_s}{2 r_a^4 \omega_E^2} \\ &= 3.294 \times 10^9 J\end{aligned}$$