Physics 3210 Spring 2019 Discussion #16 Answers

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The problem asks for the speed of the center of mass of the rod as a function of θ. Since the question isn't asking for a function of time or a parameter that requires a time to find we should try and use conservation laws first. Let's consider the conservation of energy. Initially the rod is at rest, so the starting energy is the following:

$$E_i = mg\left(\frac{L}{2}\right) \tag{1}$$

When there is a central force, like gravity or coulomb force, acting on a solid body you take the force to be acting on the center of mass. If the force of gravity acts on the center of the rod then the gravitational energy is defined by the height of the center of mass from the ground.

At a later point of time, when the rod is falling, the center of mass of the rod gains a linear kinetic energy and the rod gains a rotational kinetic energy. So the following is the energy of the rod while it is falling:

$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgy \tag{2}$$

y in (2) is the height of the center of mass of the rod. We can find a relation between y and θ using trigonometry.

$$\cos\theta = \frac{2y}{L} \tag{3}$$

We may now solve (3) for θ.

$$\theta = \arccos\frac{2y}{L} \tag{4}$$

In (2), ω is the rotational velocity of the rod. Since we have an expression for θ we may take a derivative of (4) to find ω.

$$\omega = \frac{-1}{\sqrt{1-\left(\frac{2y}{L}\right)^2}}\frac{2\dot{y}}{L} \tag{5}$$

We must figure out what $\dot{y}$ is. Let's consider conservation of momentum. In the y direction we have the force of gravity acting on the rod, so momentum isn't conserved in this direction. However, there are no forces acting on the rod in the x direction so momentum in the x direction must be conserved. Since the rod was at rest at the beginning that means that there is no velocity in the x direction. Thus, all velocity of the center of mass is in the y direction and so $\dot{y}$ is the speed of the center of mass. So we can rewrite (5) as the following:

$$\omega = \frac{-2v}{L\sqrt{1-\left(\frac{2y}{L}\right)^2}} \tag{6}$$

Now we need to find the moment of inertia I. We note from the previous argument that the rod is essentially spinning about it's center of mass. So the moment of inertia is the same as the moment of inertia of a rod in your hand out.

$$I_{rod} = \frac{1}{12}mL^2 \tag{7}$$

Plugging (6) and (7) into (2) and simplifying gives the following expression for velocity:

$$v^2 = \frac{g(L-2y)}{1+\frac{1}{3\left(1-\left(\frac{2y}{L}\right)^2\right)}} \tag{8}$$

Now we can use (3) to solve for y as a function of θ. Doing so, we obtain the following:

$$y = \frac{L\cos\theta}{2} \tag{9}$$

Plugging (9) into (8) and simplifying gives us the speed of the center of mass of the rod as a function of θ.

$$v = \sqrt{\frac{gL(1-\cos\theta)}{1+\frac{1}{3}(\csc\theta)^2}}$$