# Physics 3210 Spring 2019 Discussion \#14 Answers 

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## 1

In Lagrangian mechanics we must start with defining generalized coordinates. Luckily, the generalized coordinates are given in the problem as $x_{1}, x_{2}$, and $X$. Next we must find the kenetic energy and potential energy of the system. The kinetic energy is simply the sum of the kinetic energies of each block.

$$
\begin{equation*}
T=\frac{1}{2} m{\dot{x_{1}}}^{2}+\frac{1}{2} M \dot{X}^{2}+\frac{1}{2} m{\dot{x_{2}}}^{2} \tag{1}
\end{equation*}
$$

Now we must find the potential energies due to the springs. The following is the potential energy of the two springs:

$$
\begin{equation*}
U=\frac{1}{2} K\left(X-x_{1}\right)^{2}+\frac{1}{2} K\left(x_{2}-X\right)^{2} \tag{2}
\end{equation*}
$$

Note that to find the total stretch / compression in the springs we have to consider the displacement of both of the blocks attached to each spring. Now that we have kinetic and potential energies in terms of the generalized coordinates we may now write the Lagrangian:

$$
\begin{equation*}
\mathcal{L}=\frac{1}{2} m{\dot{x_{1}}}^{2}+\frac{1}{2} M \dot{X}^{2}+\frac{1}{2} m{\dot{x_{2}}}^{2}-\frac{1}{2} K\left(X-x_{1}\right)^{2}-\frac{1}{2} K\left(x_{2}-X\right)^{2} \tag{3}
\end{equation*}
$$

Now to recover the equations of motion for each block we will plug the Lagrangian into the Euler-Lagrange equation for each generalized coordinate.

$$
\begin{gather*}
\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial x_{1}}\right)-\frac{\partial \mathcal{L}}{\partial \dot{x_{1}}}=m \ddot{x_{1}}-K\left(X-x_{1}\right)=0  \tag{4}\\
\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial X}\right)-\frac{\partial \mathcal{L}}{\partial \dot{X}}=M \ddot{X}+K\left(X-x_{1}\right)-K\left(x_{2}-X\right)=0  \tag{5}\\
\frac{d}{d t}\left(\frac{\partial \mathcal{L}}{\partial x_{2}}\right)-\frac{\partial \mathcal{L}}{\partial \dot{x_{2}}}=m \ddot{x_{2}}+K\left(x_{2}-X\right)=0 \tag{6}
\end{gather*}
$$

If we add (4), (5), and (6) we obtain the following result:

$$
\begin{equation*}
m \ddot{x_{1}}+M \ddot{X}+m \ddot{x_{2}}=0 \tag{7}
\end{equation*}
$$

We note that (7) is the numerator of the acceleration of the center of motion. Dividing by the total mass of the system gives the following:

$$
\begin{equation*}
\frac{m \ddot{x_{1}}+M \ddot{X}+m \ddot{x_{2}}}{M+2 m}=a_{C M}=0 \tag{8}
\end{equation*}
$$

We see now that the acceleration of the center of mass is zero and thus the center of mass must move with constant acceleration.

## 2

The vectors of the normal modes can be thought of as representing the relative motion of the blocks with respect to each other. The last normal mode, $(1,-\xi, 1)$, then shows us if the first mass moves in the positive direction then the third mass will move the same amount in the same direction and the second mass will move in the opposite direction $\xi$ times the distance as the first mass. Let's assume that the first mass moves one meter and plug in the positions into the center of mass equation:

$$
\begin{equation*}
X_{C M}=\frac{m(1)+M(-\xi)+m(1)}{M+2 m}=0 \tag{9}
\end{equation*}
$$

We can then solve (9) for $\xi$. Doing so gives the following:

$$
\begin{equation*}
\xi=\frac{2 m}{M} \tag{10}
\end{equation*}
$$

## 3

Now we must find the frequencies at which each normal mode oscillates. Luckily for us, only one normal mode requires computation. The other two can be immediately obtained with logic.

The first normal mode, $(1,1,1)$, has all of the blocks moving together. If they all move together then the springs do not stretch or compress and so there is no oscillation. Thus, $\omega_{1}=0$.

The second normal mode, $(1,0,-1)$, has the two masses at the ends oscillating in opposite directions and the center mass stationary. If we look at either end mass we can note that the set up is exactly the same as if the mass and spring were attached to a wall. Thus, $\omega_{2}=\sqrt{\frac{K}{m}}$.

The last normal mode is not a trivial case, so we need to compute the frequency. To do this we will start with (5). If simple harmonic motion is achieved then the following must be true:

$$
\begin{equation*}
\ddot{X}=-\omega_{3}^{2} X \tag{11}
\end{equation*}
$$

Plugging (11) into (5) gives the following equation:

$$
\begin{equation*}
-\omega_{3}^{2} X+K\left(X-x_{1}\right)-K\left(x_{2}-X\right)=0 \tag{12}
\end{equation*}
$$

Now we will plug in the displacements of the blocks if $x_{1}$ moves one meter into (12) and solve for $\omega_{3}$. Doing so gives the following:

$$
\begin{aligned}
& -\omega_{3}^{2}\left(\frac{-2 m}{M}\right)+K\left(\frac{-2 m}{M}-1\right)-K\left(1-\frac{-2 m}{M}\right)=0 \\
& \Longrightarrow \omega_{3}=\sqrt{\frac{K}{m}\left(\frac{2 m}{M}+1\right)}
\end{aligned}
$$

