Physics 3210 Spring 2019 Discussion #13 Answers

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The problem gives us the generalized coordinates we will use to find equations of motion: r and θ. Now we need to write the kinetic and potential energies of the system in terms of the generalized coordinates. The kinetic energy of the system is the following:

$$\begin{aligned} T &= T_m + T_M \\ &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{x}^2 \end{aligned} \tag{1}$$

x is related to r with the following relation:

$$x = l - r \tag{2}$$

Taking a derivative of (2) gives us a relation between the hanging block's speed and the radial speed of the rotating block:

$$\dot{x} = -\dot{r} \tag{3}$$

Plugging (3) into (1) gives the kinetic energy in terms of only generalized coordinates:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 \tag{4}$$

The only potential energy in the system is the potential energy due to the hanging block of mass M. If we set our coordinate system such that the origin is at the hole in the table then the potential energy is the following:

$$U = -Mg(l - r) \tag{5}$$

Plugging (4) and (5) into the Lagrangian gives the following:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 + Mg(l - r) \tag{6}$$

Now that we have the Lagrangian of the system we will plug in the Lagrangian into the Euler-Lagrange equation for r and θ.

r:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = 0 \\
&\implies mr\dot{\theta}^2 - Mg - \frac{d}{dt}(m\dot{r} + M\dot{r}) = 0 \\
&\implies \ddot{r} = \frac{mr\dot{\theta}^2 - Mg}{M+m} \qquad (7)
\end{aligned}$$

θ:

$$\begin{aligned}
&\frac{\partial \mathcal{L}}{\partial \theta} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = 0 \\
&\implies 0 - \frac{d}{dt}(mr^2\dot{\theta}) = 0 \qquad (8) \\
&\implies 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0 \\
&\implies \ddot{\theta} = \frac{2\dot{r}\dot{\theta}}{r} \qquad (9)
\end{aligned}$$

Equations (7) and (9) are the equations of motion of the system in terms of r and θ. I will now derive equation (7) using Newton's Laws to demonstrate that Lagrangian mechanics and Newtonian mechanics are the same.

Considering the forces in the radial direction acting on the block of mass m, the vertical forces acting on mass M, and Newton's second law we obtain the following system of equations:

$$r : -T = m\ddot{r} - mr\dot{\theta} \qquad (10)$$

$$z : T - Mg = M\ddot{r} \qquad (11)$$

Solving (11) for the tension T and and plugging that into (10) gives the following:

$$\begin{aligned}
&m\ddot{r} - mr\dot{\theta}^2 = -M\ddot{r} - Mg \\
&\implies (M+m)\ddot{r} = mr\dot{\theta}^2 - Mg \\
&\implies \ddot{r} = \frac{mr\dot{\theta}^2 - Mg}{M+m} \qquad (12)
\end{aligned}$$

Note that (12) is the same as (7). So we may choose to solve mechanics problems with either method.

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For this part we are given the expression for angular momentum:

$$L = mr^2\dot{\theta} \tag{13}$$

If we look at equation (8) we see that angular momentum is in the equation. Plugging in (13) into (8) gives the following:

$$\frac{dL}{dt} = 0 \tag{14}$$

If the time derivative is zero then that means that the angular momentum is a constant of time.

If we solve (13) for $\dot{\theta}$ then we get the following:

$$\dot{\theta} = \frac{L}{mr^2} \tag{15}$$

Plugging (15) into (7) gives the following expression for radial acceleration in terms of angular momentum:

$$\ddot{r} = \frac{1}{M+m}\left(\frac{L^2}{mr^3} - Mg\right) \tag{16}$$

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If the block of mass m is moving with uniform circular motion than the radial acceleration must be zero. Setting $\ddot{r}$ in (16) to zero and solving for r gives us the radius at which uniform circular motion is achieved.

$$\begin{aligned}
&0 = \frac{1}{M+m}\left(\frac{L^2}{mr^3} - Mg\right) \\
&\implies \frac{L^2}{mr^3} = Mg \\
&\implies \frac{mr^3}{L^2} = \frac{1}{Mg} \\
&\implies r_0 = \left(\frac{L^2}{mMg}\right)^{\frac{1}{3}}
\end{aligned}$$