# Physics 3210 Spring 2019 Discussion #1 Answers

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## 1

### 1.1

First, let's find $\overrightarrow{A} \times \overrightarrow{B}$. To find $\overrightarrow{A} \times \overrightarrow{B}$ we will compute the determinant of the matrix with the standard Cartesian orthonormal basis as the first row, the coefficients of $\overrightarrow{A}$ as the second row, and the coefficients of $\overrightarrow{B}$ as the last row:

$$\begin{aligned}
\overrightarrow{A} \times \overrightarrow{B} &= Det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 2 & 3 & -4 \\ -6 & -4 & 1 \end{bmatrix} \\
&= \hat{x} * \begin{bmatrix} 3 & -4 \\ -4 & 1 \end{bmatrix} - \hat{y} * \begin{bmatrix} 2 & -4 \\ -6 & 1 \end{bmatrix} + \hat{z} * \begin{bmatrix} 2 & 3 \\ -6 & -4 \end{bmatrix} \\
&= (3 * 1 - (-4)(-4)) * \hat{x} - (2 * 1 - (-4)(-6)) * \hat{y} + (2(-4) - 3(-6)) * \hat{z} \\
&= -13\hat{x} + 22\hat{y} + 10\hat{z}
\end{aligned}$$

To compute the the angle $\theta$ between $\overrightarrow{A} \times \overrightarrow{B}$ and $\overrightarrow{C}$ we will use the following definition of the dot product:

$$\begin{aligned}
(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C} &= |\overrightarrow{A} \times \overrightarrow{B}||\overrightarrow{C}| \cos(\theta) \\
\iff \theta &= \arccos\left(\frac{(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C}}{|\overrightarrow{A} \times \overrightarrow{B}||\overrightarrow{C}|}\right) \qquad (1)
\end{aligned}$$

Now we compute the dot product of $(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C}$:

$$\begin{aligned}
(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C} &= (-13\hat{x} + 22\hat{y} + 10\hat{z}) * (1\hat{x} - 1\hat{y} + 1\hat{z}) \\
&= 1(-13) + 22(-1) + 10(1) \\
&= -25
\end{aligned}$$

and the magnitudes of $\overrightarrow{A} \times \overrightarrow{B}$ and $\overrightarrow{C}$:

$$
\begin{aligned}
|\overrightarrow{A} \times \overrightarrow{B}| &= \sqrt{(-13)^2 + (22)^2 + (10)^2} \\
&= \sqrt{753}
\end{aligned}
$$

$$
\begin{aligned}
|\overrightarrow{C}| &= \sqrt{(1)^2 + (-1)^2 + (1)^2} \\
&= \sqrt{3}
\end{aligned}
$$

Now we simply plug these values into (1) to find the angle between $\overrightarrow{A} \times \overrightarrow{B}$ and $\overrightarrow{C}$:

$$
\begin{aligned}
\theta &= \arccos\left(\frac{(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C}}{|\overrightarrow{A} \times \overrightarrow{B}||\overrightarrow{C}|}\right) \\
&= \arccos\left(\frac{-25}{\sqrt{753} * \sqrt{3}}\right) \\
&\approx 2.125 \text{ radians or } 121.735 \text{ degrees}
\end{aligned}
$$

## 1.2

To find the magnitude of $\overrightarrow{A} \times \overrightarrow{B}$ in the direction of $\overrightarrow{C}$ we will use the dot product.

First we must normalize $\overrightarrow{C}$. We can do this quickly by dividing $\overrightarrow{C}$ by its own magnitude:

$$
\begin{aligned}
\hat{C} &= \frac{\overrightarrow{C}}{\sqrt{3}} \\
&= \frac{1}{\sqrt{3}}\hat{x} - \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{x}
\end{aligned}
$$

Now we simply take the dot product of $\overrightarrow{A} \times \overrightarrow{B}$ and $\hat{C}$ to find the magnitude of $\overrightarrow{A} \times \overrightarrow{B}$ in the direction of $\overrightarrow{C}$:

$$
\begin{aligned}
(\overrightarrow{A} \times \overrightarrow{B}) \cdot \hat{C} &= (-13\hat{x} + 22\hat{y} + 10z) \cdot (\frac{1}{\sqrt{3}}\hat{x} - \frac{1}{\sqrt{3}}\hat{y} + \frac{1}{\sqrt{3}}\hat{x}) \\
&= (-13)(\frac{1}{\sqrt{3}}) + (22)(-\frac{1}{\sqrt{3}}) + (10)(\frac{1}{\sqrt{3}}) \\
&= \frac{-25}{\sqrt{3}} \\
&\approx -14.434
\end{aligned}
$$

# 2

## 2.1

To find a vector perpendicular to the specified plane we can find two vectors parallel to the plane and then take their cross product. Since the plane contains the origin, any solution to the plane equation will be a vector parallel to the plane. We will pick the points $(1, 1, 1)$ and $(-1, 0, 2)$ to be our parallel vectors. Taking the cross product gives:

$$
\begin{aligned}
\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} &= \hat{x} * \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \hat{y} * \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} + \hat{z} * \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \\
&= (1 * 2 - 1 * 0) * \hat{x} - (2 * 1 - 1(-1)) * \hat{y} + (1 * 0 - 1(-1)) * \hat{z} \\
&= 2\hat{x} - 3\hat{y} + \hat{z}
\end{aligned}
$$

We can also use the gradient to find a perpendicular vector. We start by first defining a function that matches the plane equation:

$$f(x, y, z) = 2x - 3y + z \tag{2}$$

The gradient of a function points in the direction of greatest ascent, which is perpendicular to any level curve $f(x, y, z) = k$ where $k \in \mathbb{R}$. Note that choosing $k = 0$ recovers the plane in question. So the gradient of (2) will give a vector perpendicular to the plane:

$$
\begin{aligned}
\nabla(f(x, y, z)) &= \frac{\partial}{\partial x}(2x - 3y + z)\hat{x} + \frac{\partial}{\partial y}(2x - 3y + z)\hat{y} + \frac{\partial}{\partial z}(2x - 3y + z)\hat{z} \\
&= 2\hat{x} - 3\hat{y} + \hat{z}
\end{aligned}
$$

## 2.2

We have already expressed a vector perpendicular to $f(x, y, z) = 0$ in Cartesian coordinates (we will call this vector $\overrightarrow{V}$ for convenience).

First, let's convert from Cartesian coordinates to cylindrical coordinates. The following equations give the conversions from Cartesian to cylindrical:

$$\rho = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \quad \text{if } x > 0$$

$$z = z$$

so, in cylindrical coordinates:

$$
\begin{aligned}
\overrightarrow{V} &= (\rho, \theta, z) \\
&= \left(\sqrt{2^2 + (-3)^2}, \arctan\left(\frac{-3}{2}\right), 1\right) \\
&\approx (3.606, -0.983, 1)
\end{aligned}
$$

Now we will convert to spherical coordinates. The following equations give conversions from Cartesian to spherical:

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \quad \text{if } z > 0$$

$$\theta = \arctan\left(\frac{y}{x}\right) \quad \text{if } x > 0$$

So, in spherical coordinates:

$$
\begin{aligned}
\overrightarrow{V} &= (\rho, \theta, \phi) \\
&= \left(\sqrt{2^2 + (-3)^2 + 1^2}, \arctan\left(\frac{\sqrt{2^2 + (-3)^2}}{1}\right), \arctan\left(\frac{-3}{2}\right)\right) \\
&= \left(\sqrt{14}, \arctan\left(\sqrt{13}\right), \arctan\left(\frac{-3}{2}\right)\right) \\
&\approx (3.712, 1.300, -0.983)
\end{aligned}
$$

Just a quick last note, both sets of coordinate conversion equations can be obtained by picking a general point, obtaining that point with Cartesian and cylindrical/spherical coordinates, and then analyzing the relations between those coordinates.